A Glimpse of the World of Topos Theory

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The following is an introduction to some basic features of topos theory. Following a brief review of the basic notions from category theory I will develop the definition of a topos, from the point of view of it as a generalization of the category of sets and the basics of its connection to logic. Then I will provide a wide variety of examples of toposes, including categories of monoid actions and categories of sheaves over a topological space. I will then shift to a discussion of the important example of the topos **Set** of sets and I will discuss the characteristic features of the category, including its characterization as a topos of constant sets. I will also consider properties of toposes that are **Set**-like, but not necessarily equivalent to the topos of sets **Set**. I will then examine some of the basic aspects of the relationship between logic and toposes. This leads into a discussion of the notions of a *local language* and a *local set theory*. These notions lead to a characterization of the category of sets that is generated by an arbitrary topos, clarifying how a topos is a generalized category of sets. In this context I briefly discuss some of the important results about local set theories. Finally, to complete the characterization of the category of sets as a topos and to examine how number systems are defined in a topos, I briefly describe the generalization of the arithmetic of the natural numbers and the real numbers to toposes.

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1 Review of Concepts from Category Theory

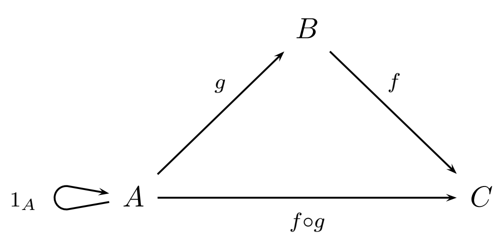
Some background in category theory is being assumed for this presentation, but this section provides a brief review the basic notions coming from category. Recall that a category is a collection of objects A, B, C and arrows f, g, h that satisfies three simple axioms:

Definition 1 A *category* is a collection of objects A, B, C, *etc.* and arrows f, g, h, *etc.* such that the following axioms hold:

1. Each object A has an identity arrow 1_A that goes from A to A;
2. For any two composable arrows f and g, there is a composite arrow from the domain of g, $dom(g)$, to the codomain of f, $cod(f)$;
3. Composition of arrows is associative.

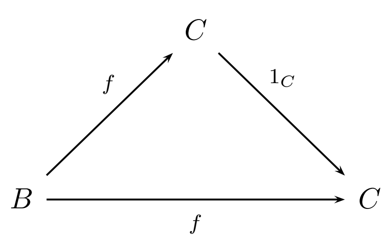
Since composition of arrows is associative, the composition of several arrows can be written unambiguously without brackets.

The axioms may be written in the form of a set of equations or, more attractively, in the form of a set of commutative diagrams. The axioms displayed in these two ways are the following:

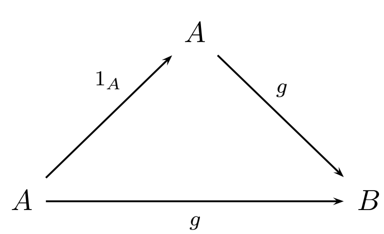


$$dom(f \circ g) = dom(g) \quad \text{and} \quad cod(f \circ g) = cod(f)$$

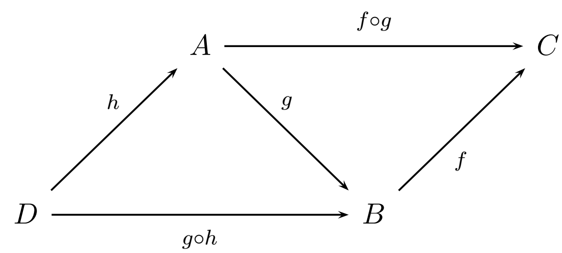
$$1_A \colon A \longrightarrow A \quad dom(1_A) = A \quad \text{and} \quad cod(1_A) = A$$



$$1_C \circ f = f$$



$$g \circ 1_A = g$$



$$(f \circ g) \circ h = f \circ (g \circ h)$$

Definition 2 Let collection $\mathbf{C}$ be a category, then $\mathbf{C}(A, B)$ is the collection of all arrows from A to B in $\mathbf{C}$.

Since in many concrete categories arrows are homomorphisms between mathematical structures of a given kind, $\mathbf{C}(A, B)$ is sometimes call a *hom-set*.

There are three special kinds of arrows that can exist in a category. They are the categorical versions of monomorphisms, *i.e.* injective homomorphisms, epimorphisms, *i.e.* surjective homomorphisms, and isomorphisms. The generalization of a monomorphism is a monic arrow.

Definition 3 An arrow $f\colon A \to B$ is *monic* if it is 'left-cancellable,' *i.e.* if for any pair of arrows $x, y\colon T \to A$ to A,

$$f \circ x = f \circ y \Longrightarrow x = y.$$

A monic arrow can be denoted using a tailed arrow, *i.e.* as $f\colon A \rightarrowtail B$. The dual concept to a monic arrow is an epic arrow, the generalization of an epimorphism.

Definition 4 An arrow $g\colon A \to B$ is *epic* if it is 'right-cancellable,' *i.e.* if for any pair of arrows $x, y\colon B \to T$ from B,

$$x \circ g = y \circ g \Longrightarrow x = y.$$

An epic arrow can be denoted using a double headed arrow, *i.e.* as $f\colon A \twoheadrightarrow B$. The generalization of an isomorphism is an iso arrow.

Definition 5 An arrow $f\colon A \to B$ is *iso* if there is an arrow $g\colon B \to A$ such that

$$f \circ g = 1_B \quad \text{and} \quad g \circ f = 1_A.$$

In such a case g is called the *inverse* of f and is denoted f^{-1}.

An iso arrow can be denoted as $f\colon A \xrightarrow{\sim} B$.

There are also 'split' versions of epic and monic arrows.

Definition 6 An arrow $f\colon A \to B$ is *split monic* if there is an arrow $g\colon B \to A$ such that $g \circ f = 1_A$. Such an arrow g is called a *right inverse*.

Thus, an arrow is split monic if it has a right inverse. The dual notion is a split epic arrow.

Definition 7 An arrow $f\colon A \to B$ is *split epic* if there is an arrow $g\colon B \to A$ such that $f \circ g = 1_B$.

Thus, an arrow is split epic if it has a left inverse. Any arrow that is monic and split epic is iso, and any arrow that is epic and split monic is iso. An arrow that is epic and monic is not always iso, but it every topos, each epic monic arrow is iso.

There are a wide variety of categorical structures that exist in many categories. The simplest are initial and terminal objects. Recall that

Definition 8 a *terminal object* is an object 1 in a category $\mathbf{C}$ such that there is a unique arrow to it from any object of $\mathbf{C}$.

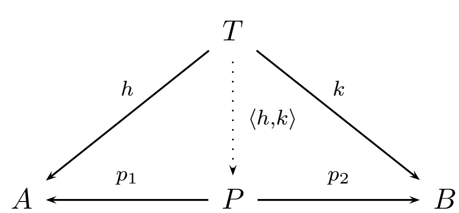
If a category has a terminal object, then this provides a way of talking about *elements* of an object A. The elements of A are the arrows $1 \to A$. In set theory, the elements of a set X are in 1-1 correspondence with the arrows from the set $1 = \{0\}$ to X. Such arrows are sometimes called *global objects*, which reflects the origin of the notion as a global section of a sheaf. Another kind of element that can be considered is a *generalized element* of an object. A generalized element of an object A is just an arrow from some object T to A. Where global elements can be thought of as single, and constant, elements, a generalized element could be thought of as a *variable* element of an object. The dual notion a terminal object is an *initial object*, which is an object 0 in a category $\mathbf{C}$ such that there is a unique arrow from it to any object of $\mathbf{C}$.

A more complex structure is a product.

Definition 9 Given two objects A and B, a *product* of A and B consists of an object P and two arrows p_1 and p_2 such that

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

and they satisfy the following universal mapping property: Let T be any object together with two arrows h and k such that $A \xleftarrow{h} T \xrightarrow{k} B$. Then there is a unique arrow $\langle h, k\rangle\colon T \longrightarrow P$ such that the following diagram commutes:



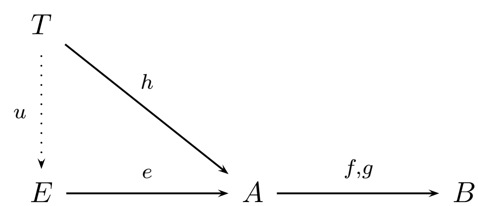
In equations, $p_1 \circ \langle h, k\rangle = h$ and $p_2 \circ \langle h, k\rangle = k$. p_1 and p_2 are called the *projection arrows* of the product P, p_1, p_2.

Given two products $A \times B, p_1, p_2$ and $A' \times B', p_1, p_2$ and arrows $f\colon A \to A'$ and $g\colon A \to B$ there is a unique arrow $f \times g$ from $A \times B$ to $A' \times B'$ that commutes with the arrows f and g. This is called the *product arrow* of f and g.

The structure dual to a product is a *coproduct*, which is defined by reversing the arrows in the definition of the product. A coproduct of two objects A and B is denoted $A+B$, i_1, i_2, where the arrows $i_1\colon A \to A+B$ and $i_2\colon B \to A+B$ are called *inclusion arrows*. The unique map in the definition of the coproduct is denoted $[f,g]$. Analogously to the definition of the product of two arrows, one may define the sum $f+g$ of two arrows.

Another important structure is an equalizer.

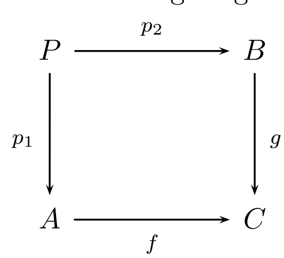
Definition 10 Let $\mathcal{C}$ be a category and $f, g\colon A \longrightarrow B$ a parallel pair of arrows of $\mathcal{C}$. An *equalizer* of f and g is an arrow $e\colon E \longrightarrow A$ that equalizes f and g, *i.e.* $f \circ e = g \circ e$, and for any arrow $h\colon T \longrightarrow A$ that equalizes f and g, there is a unique arrow u making the following diagram commute:



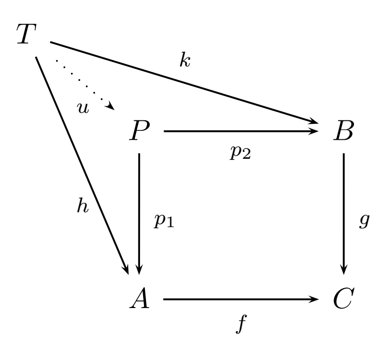
It is not too difficult to show that all equalizers are monic. Equalizers $e\colon E \rightarrowtail A$ for $f, g\colon A \to B$ can be thought of as the part of A on which the arrows f and g agree. The dual notion of a *coequalizer* is defined by reversing the arrows in the definition of an equalizer. It is not too difficult, then, to show that all coequalizers are epic.

The last specific type of structure, and its dual, we will consider is a pullback.

Definition 11 A *pullback* of a corner of arrows $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ consists of an object P and two arrows such that the following diagram commutes:



and such that for any object T and arrows $h\colon T \longrightarrow A$ and $k\colon T \longrightarrow B$ such that the outer square below commutes, *i.e.* $f \circ h = g \circ k$, then there is a unique $u\colon T \longrightarrow P$ making the entire diagram commute, *i.e.* also $h = p_1 \circ u$ and $k = p_2 \circ u$:



In such a case we may call p_1 the *pullback of g along f* and p_2 the *pullback of f along g*.

In set theory many common structures can be seen to be pullbacks. This includes the intersection of two sets, the inverse image of a subset of a set and the kernel of a homomorphism. The notion of a pullback also appears in differential geometry, where one can 'pull back' structures on one manifold Y to another X along a smooth map $f\colon X \to Y$. The dual notion of a *pushout* is defined by reversing the arrows in the definition of a pullback. The union of two sets can be seen to be a pushout, and pushouts appear similarly in differential geometry, where they are called *pushforwards*.

All of the structures we have considered explicitly are examples of limits, and their duals are all examples of colimits. Thus, these notions encompass all those structures mentioned here and many others. To define limits we first need the notion of a diagram and a cone. Let $\mathbf{C}$ be a category.

Definition 12 A *diagram* D in $\mathbf{C}$ is a directed graph where the vertices are objects of $\mathbf{C}$ and the edges are arrows of $\mathbf{C}$.

Definition 13 Let D be a diagram in $\mathbf{C}$. Then a *cone* over D consists of an object C together with an arrow $p_i\colon C \longrightarrow A_i$ to each object A_i of D such that if $f\colon A_i \longrightarrow A_j$ is an arrow of D, then $f \circ p_i = p_j$.

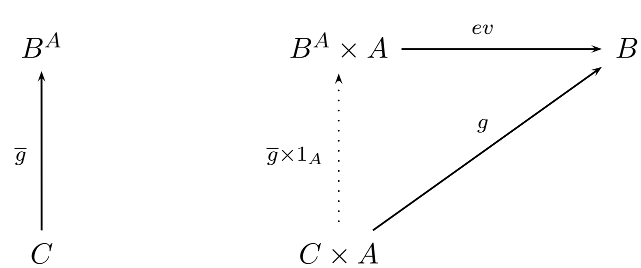
Then we have the following definition of a limit:

Definition 14 Then a *limit* L, p_i for a diagram D is a cone over a diagram D in $\mathbf{C}$ such that for each cone C, q_i over D, there is a unique arrow u from C to L such that $p_i \circ u = q_i$ for all i. The arrows p_i of L are called *projection arrows*.

If a category has a limit for a diagram D, then it is *unique up to isomorphism*, in the sense that any two limits for the same diagram are isomorphic and if there is an iso arrow from a limit to a cone, then that cone is also a limit. Thus, all of the structures we have considered explicitly are unique up to isomorphism. The dual notion of a *colimit* is defined by reversing the arrows in the definition of a limit, which requires the notion of a *cocone*, the dual to a cone. Just as for limits, if a category has a colimit for a diagram then it is unique up to isomorphism. Thus, the duals to all the structures we have defined explicitly are unique up to isomorphism.

An important special kind of object that categories may obtain is an exponential object.

Definition 15 Given objects A and B an *exponential* of B by A consists of an object B^A and an arrow $ev\colon B^A \times A \longrightarrow B$, called an *evaluation arrow*, such that for any object C and arrow $g\colon C \times A \longrightarrow B$ there is a unique arrow $\overline{g}\colon C \longrightarrow B^A$ such that the following diagram commutes:



An exponential object B^A can be thought of as the object of all arrows from A to B. In set theory, the exponential set X^Y is the set of all functions from Y to X.

Definition 16 A category that has limits for all finite diagrams is said to have *all finite limits*.

It can be shown that a category has all finite limits iff it has a terminal object and pullbacks for all corners of arrows. It can also be shown that a category has all finite limits if it has a terminal object, binary products and equalizers for every parallel pair of arrows. A category that has all finite limits and exponentials is called *cartesian closed*.

Definition 17 A category is called *cartesian closed* if is has all finite limits and an exponential for each pair of objects.

Part of the definition of a topos is that it is a cartesian closed category.

Definition 18 A category that has colimits for all finite diagrams is said to have *all finite colimits*.

Thus, a category has all finite colimits iff it has an initial object and pushouts for all corners of arrows, and iff it has an initial object, binary coproducts and coequalizers for every parallel pair of arrows. Toposes also have all finite colimits, but as we shall see this need not be included in the definition.

The last collection of notions that we need all involve relations between categories. The first is the notion of a structure preserving map between two categories.

Definition 19 A *functor* from a category $\mathbf{C}$ to a category $\mathbf{D}$, written $\boldsymbol{F}\colon \mathbf{C} \longrightarrow \mathbf{D}$, maps each object A of $\mathbf{C}$ to an object $\boldsymbol{F}A$ of $\mathbf{D}$, and each arrow $f\colon A \longrightarrow B$ of $\mathbf{C}$ to an arrow $\boldsymbol{F}f$ of $\mathbf{D}$, such that the following conditions are satisfied:

1. Given $f\colon A \longrightarrow B$ of $\mathbf{A}$, we have that $\boldsymbol{F}f\colon \boldsymbol{F}A \longrightarrow \boldsymbol{F}B$ (preservation of domains and codomains);
2. For any A of $\mathbf{C}$, $\boldsymbol{F}(1_A) = 1_{\boldsymbol{F}A}$ (preservation of identities);
3. If f and g are composable arrows in $\mathbf{C}$, then $\boldsymbol{F}(g \circ f) = \boldsymbol{F}g \circ \boldsymbol{F}f$, which is the composite of $\boldsymbol{F}f$ and $\boldsymbol{F}g$ in $\mathbf{D}$ (preservation of composition).

There are two important functions that may be obtained from any functor.

Definition 20 Let $\boldsymbol{F}$ be a functor from $\mathbf{C}$ to $\mathbf{D}$. Then the function on the collection of objects of $\mathbf{C}$ such that $A \mapsto \boldsymbol{F}A$ is called the *object function* of $\boldsymbol{F}$. The function on the collection of arrows of $\mathbf{C}$ such that $f \mapsto \boldsymbol{F}f$ is called the *arrow function* of $\boldsymbol{F}$.

Four important classes of functors are defined in terms of whether these functions are injective or surjective.

Definition 21 Let $\boldsymbol{F}\colon \mathbf{C} \longrightarrow \mathbf{D}$ be a functor. Then we define the following four classes of functors (recall that $\mathbf{C}(A, B)$ is the collection of all arrows from A to B in $\mathbf{C}$):

- $\boldsymbol{F}$ is *full* if for any objects A and B of $\mathbf{C}$, $\boldsymbol{F}$ maps $\mathbf{C}(A, B)$ onto $\mathbf{D}(\boldsymbol{F}A, \boldsymbol{F}B)$, *i.e.* the restriction of the *arrow* function of $\boldsymbol{F}$ to any $\mathbf{C}(A, B)$ is onto $\mathbf{D}(\boldsymbol{F}A, \boldsymbol{F}B)$;
- $\boldsymbol{F}$ is *faithful* if for any objects A and B of $\mathbf{C}$, $\boldsymbol{F}$ is one-to-one on $\mathbf{C}(A, B)$, *i.e.* the restriction of the *arrow* function of $\boldsymbol{F}$ to any $\mathbf{C}(A, B)$ is injective;

- $\boldsymbol{F}$ is *dense* if for any $\mathbf{D}$-object B, there is a $\mathbf{C}$ object A such that $\boldsymbol{F}A \simeq B$, *i.e.* the *object* function is 'onto up to isomorphism';
- $\boldsymbol{F}$ is an *embedding* if it is faithful and the *object* function is injective, *i.e.* $\boldsymbol{F}A = \boldsymbol{F}B \Longrightarrow A = B$.

The following important notion is the precise notion of a structure preserving mapping between two functors.

Definition 22 Given a parallel pair of functors $\boldsymbol{F}, \boldsymbol{G} \colon \mathbf{C} \to \mathbf{D}$, a *natural transformation* from $\boldsymbol{F}$ to $\boldsymbol{G}$ is a family of arrows $\nu_A \colon \boldsymbol{F}A \to \boldsymbol{G}A$, one for each object A of $\mathbf{C}$, such that for every arrow $f \colon A \to A'$ in $\mathbf{C}$, the following diagram commutes:

$$
\begin{array}{ccccc}
A & \qquad & \boldsymbol{F}A & \xrightarrow{\nu_A} & \boldsymbol{G}A \\
{\scriptstyle f}\downarrow & & {\scriptstyle \boldsymbol{F}f}\downarrow & & \downarrow{\scriptstyle \boldsymbol{G}f} \\
A' & & \boldsymbol{F}A' & \xrightarrow[\nu_{A'}]{} & \boldsymbol{G}A'
\end{array}
$$

We write $\nu \colon \boldsymbol{F} \to \boldsymbol{G}$ for the natural transformation and we call the arrows ν_A the *components* of the natural transformation ν.

Definition 23 A natural transformation such that every component is an iso arrow is called a *natural isomorphism*.

Given three functors $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ and two natural transformations $\nu \colon \boldsymbol{F} \to \boldsymbol{G}$ and $\varphi \colon \boldsymbol{G} \to \boldsymbol{H}$, then one may obtain a natural transformation $\varphi \circ \nu \colon \boldsymbol{F} \to \boldsymbol{H}$ by composing each of the components ν_A and φ_A:

$$
\begin{array}{ccccccc}
A & \qquad & \boldsymbol{F}A & \xrightarrow{\nu_A} & \boldsymbol{G}A & \xrightarrow{\varphi_A} & \boldsymbol{H}A \\
{\scriptstyle f}\downarrow & & {\scriptstyle \boldsymbol{F}f}\downarrow & & {\scriptstyle \boldsymbol{G}f}\downarrow & & \downarrow{\scriptstyle \boldsymbol{H}f} \\
A' & & \boldsymbol{F}A' & \xrightarrow[\nu_{A'}]{} & \boldsymbol{G}A' & \xrightarrow[\varphi_{A'}]{} & \boldsymbol{H}A'
\end{array}
$$

This enables the construction of categories of functors.

Definition 24 Given two categories $\mathbf{A}$ and $\mathbf{B}$, a *functor category* $\mathbf{B}^{\mathbf{A}}$ is a category of functors from $\mathbf{A}$ to $\mathbf{B}$ and natural transformations between functors. The identity natural transformation for any given functor $\mathbf{F}$ is $1_{\boldsymbol{F}} \colon \boldsymbol{F} \to \boldsymbol{F}$, which has components $1_{\boldsymbol{F}A}$, and composites of natural transformations are defined as above.

Definition 25 An *equivalence situation* is a pair of functors $\boldsymbol{F} \colon \mathbf{X} \to \mathbf{A}$ and $\boldsymbol{G} \colon \mathbf{A} \to \mathbf{X}$ such that there are natural isomorphisms $\varphi \colon \mathbf{1}_{\boldsymbol{X}} \to \boldsymbol{GF}$ and $\psi \colon \mathbf{1}_{\boldsymbol{A}} \to \boldsymbol{FG}$

It can be shown that in an equivalence situation, both functors $\boldsymbol{F}$ and $\boldsymbol{G}$ are full and faithful. In an equivalence situation, every object X of $\mathbf{X}$ is isomorphic to an object in the image of $\boldsymbol{G}$, *e.g.* $\boldsymbol{GF}X$, and each object A of $\mathbf{A}$ is isomorphic to an object in the image of $\boldsymbol{F}$, *e.g.* $\boldsymbol{FG}X$.

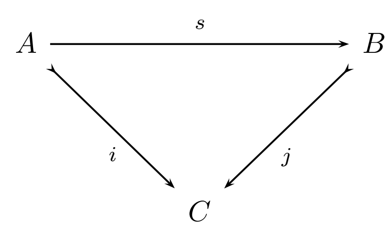
Definition 26 An *adjunction* $\langle \boldsymbol{F}, \boldsymbol{G}, \eta, \varepsilon \rangle$ is a pair of functors $\boldsymbol{F}$ and $\boldsymbol{G}$ and natural transformations η and ε, such that

$$\boldsymbol{F}\colon \mathbf{X} \to \mathbf{A}, \quad \boldsymbol{G}\colon \mathbf{A} \to \mathbf{X}, \quad \eta\colon \mathbf{1}_{\boldsymbol{F}} \to \boldsymbol{GF}, \quad \varepsilon\colon \boldsymbol{FG} \to \mathbf{1}_{\boldsymbol{A}}$$

and $\varepsilon_{\boldsymbol{F}} \circ \boldsymbol{F}\eta = 1_{\boldsymbol{F}}$ and $\boldsymbol{G}\varepsilon \circ \eta_{\boldsymbol{G}} = 1_{\boldsymbol{G}}$, which is to say that $\varepsilon_{\boldsymbol{F}X} \circ \boldsymbol{F}\eta_X = 1_{\boldsymbol{F}X}$ and $\boldsymbol{G}\varepsilon_A \circ \eta_{\boldsymbol{G}A} = 1_{\boldsymbol{G}A}$.

Definition 27 Let $\mathbf{C}$ be a category, and A an object of $\mathbf{C}$. A *subobject* of A is a monic arrow $i\colon S \rightarrowtail A$ from some object S to A.

Definition 28 Let $i\colon A \rightarrowtail C$ and $j\colon B \rightarrowtail C$ be monic arrows. Then i is *included* in j, written $i \subseteq j$, if i factors through j, *i.e.* there is some $s\colon A \longrightarrow B$ such that the following diagram commutes:



Definition 29 Let $i\colon A \rightarrowtail C$ and $j\colon B \rightarrowtail C$ be subobjects of C. Then i is *equivalent* to j, written $i \equiv j$, provided that $i \subseteq j$ and $j \subseteq i$. In such a case the arrow from A to B is iso and so we may also say that i is *isomorphic* to j and write $i \simeq j$.

2 The Definition of a Topos

Now that we have reviewed many of the important notions coming from category theory, we are nearly in a position to define a topos. I will provide the remaining concepts we need shortly, but it will perhaps be useful to start by giving a picture of what toposes are and involve. Toposes are often described as categories of "generalized sets." This is because the objects in a topos behave in many ways like sets in the universe V of sets. What I mean by 'behave' here is that toposes are analogous to the universe of sets in various ways. Some of these are the following:

1. Unlike categories in general, toposes are equipped with a (internal) logical language which enables the objects of the topos to be talked about and thought of as sets. This is because all toposes have an object of *truth values* Ω, and arrows from objects A to Ω can be interpreted as propositions in a (local) language, just as in set theory where functions from a set X to 2 correspond to propositions about the set X;

2. Unlike categories in general, all toposes have a terminal object, so it is possible to talk about (global) elements of an object;

3. Unlike categories in general, it is always possible to perform the common sorts of operations that are possible with sets. Let $\mathcal{E}$ be a topos. Then, we have the following:

   (a) Given objects A and B, their product $A \times B$ is in $\mathcal{E}$, so we can talk about products and powers A^n of objects;

(b) Given objects A and B, their coproduct $A + B$ is in $\mathcal{E}$;

(c) Given two parallel arrows, $f, g\colon A \to B$, they have an equalizer $e\colon E \rightarrowtail A$ in $\mathcal{E}$, so we can talk about the part on which two parallel arrows agree;

(d) Given two parallel arrows, $f, g\colon A \to B$, they have a coequalizer $q\colon B \twoheadrightarrow Q$ in $\mathcal{E}$, which (together with other topos axioms) enables us to construct quotient objects;

(e) Given any corner of arrows, $f\colon A \to C$, $g\colon B \to C$, their pullback P, p_1, p_2 is in $\mathcal{E}$, which enables us to talk about intersections, inverse images, kernels and all sorts of other things;

(f) Given any two objects A and B, their exponential B^A is in $\mathcal{E}$, which enables us to talk about all the 'maps' from one object to another.

In fact, toposes have all finite limits and all finite colimits. Thus, a wide variety of different kinds of mathematical constructions are possible.

4. Unlike categories in general, it is possible to pick out parts S of a given object A and to talk about *all* the parts of an object in a topos; just like we have the power set $\mathcal{P}(X) \simeq 2^X$ the subsets of a set X in set theory, we have a *power object* Ω^A of an object A, the (global) elements of which are (externally) in 1-1 correspondence with the parts of A.

Thus, in a topos, we have a set-theoretic language, a specification of all the parts of an object, and a great variety of different mathematical operations and constructions are possible. This fleshes out a bit the claim that toposes are like generalized categories of sets. This enables toposes to be thought of as alternative universes in which to do mathematics, which has inspired a philosophy surrounding topos theory, which is playfully called "toposophy." In the words of Bell (2005), the "chief tenet [of this philosophy] is the idea that, like a model of set theory, any topos may be taken as an autonomous universe of discourse or "world" in which mathematical concepts can be interpreted and constructions performed." (p. 284) Thus, mathematical statements are not seen to be true *simpliciter* (*i.e.* true of sets), but rather as true *relative* to a mathematical framework in which they are interpreted.[1]

With a perspective now on what toposes are all about, we are ready to better appreciate the definition. First of all, any topos is a *cartesian closed category*. This is what can be seen to enable much of the wide variety of mathematical constructions possible in a topos. Not only this, however, since, for example, this does not provide one with a logical language or the ability to talk about all the subobjects of an object. What brings toposes to life, as it were, is a that they contain a *subobject classifier*.

As pointed out above, the category of sets **Set** contains an object of truth values $2 = \{0, 1\}$, and functions to 2 from a given set X correspond to subsets Y of X. The subset Y picked out by a function $\chi_Y\colon X \to 2$ is the set of all elements that are mapped to 1, which may be thought of as the truth value *true*. Thus, given $a \in X$, we have that $a \in Y$ is true if and only if $\chi_Y(a) = 1$. The functions $\chi_{(\cdot)}$ are called *characteristic functions*. Each such function is characteristic of a particular subset of a set. You will see that the set of

[1]This makes propositions that are true in any topos of particular interest, which turn out to be the propositions of constructive mathematics.

characteristic functions is just the exponential set 2^X, and that the elements of this set are in 1-1 correspondence with subsets of X, *i.e.*

$$2^X \simeq \mathcal{P}(X).$$

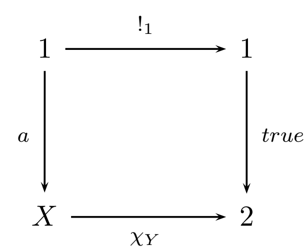
Now, to make the transition to categories, we need to reconstruct this in terms of functions. An element $a \in X$ will thus be thought of as a function $a\colon\ 1 \to X$. If we want to do everything in terms of functions, given $Y \subseteq X$ and $a \in X$, how do we express the propostion '$a \in Y$ is true?' It is important to be clear about the difference between '$a \in Y$' and '$a \in Y$ is true.' The former is a proposition, which could be true or false, and the latter is an expression that the proposition takes the truth value 'true.' Thus, we are asking how to express '$a \in Y$ is true' entirely in terms of functions!

First, we can label the truth values. Let '*true*' be the arrow

$$\mathit{true}\colon\ 1 \to 2$$

defined by $\mathit{true}(0) = 1$, with the arrow *false* being the other possibility. Now, recall that it is true that $a \in Y$ if and only if $\chi_Y(a) = 1$. Thus, we may see that $\ulcorner \chi_Y(a) \urcorner$ corresponds to the proposition '$a \in Y$,' and the expression $\ulcorner \chi_Y(a) = 1 \urcorner$ corresponds to the statement that '$a \in Y$' takes the truth value 1, *i.e.* that a actually is a member of Y. Thus, to express the membership of a in Y entirely in terms of functions we need to express $\chi_Y(a) = 1$ entirely in terms of functions.

Consider the following diagram:

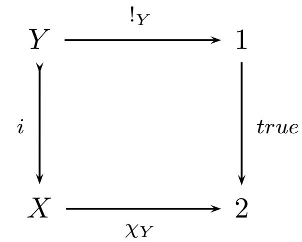


The composite of the two arrows on the lower left is $\chi_Y \circ a = \chi_Y(a)$ and the composite of those on the upper right is $\mathit{true} \circ !_1$, which you will notice is just *true*. Now, suppose that it is true that $a \in Y$, then no matter which path we follow, *i.e.* the upper right or lower left, we get the same result. This is just to say that the diagram commutes! Thus, the expression that $a \in Y$ is true is just

$$\chi_Y(a) = \mathit{true}.$$

This shows how we can express the truth of a proposition, in this case $\ulcorner \chi_Y(a) \urcorner$, which we have seen can be thought of as the proposition '$a \in Y$,' in terms of a commutative diagram. This illustrates how the truth value set 2 can be used to express the truth of propositions, *i.e.* functions from a set to 2, using *only* functions.

One more thing before we define the subobject classifier. Notice the following. Let $Y \subseteq X$, let $i\colon Y \rightarrowtail X$ be the inclusion arrow, and then consider the following diagram:



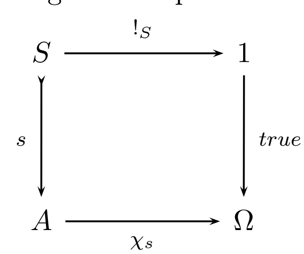
First of all notice that for each element $a\colon 1 \to Y$ of Y,

$$(\chi_Y \circ i)(a) = 1 = (true \circ !_Y)(a) = true_Y(a),$$ [2]

which is seen to express the commutativity of the diagram.[3] You will also notice that Y is the inverse image of 1 along χ_Y, *i.e.* Y is the part of X that gets mapped to 1 by χ_Y. Thinking just in terms of functions, the inclusion arrow $i\colon Y \rightarrowtail X$ is the inverse image of *true* along χ_Y. Recalling the relation of inverse images to pullbacks, it will not be too surprising that the above diagram is a pullback. That this diagram is a pullback, together with the fact that each subset Y of X has a unique characteristic function χ_Y, is an expression of the fact that the truth value set 2 together with the function *true* is a *subobject classifier.*

We are now ready to define a subobject classifier.

Definition 30 Let $\mathbf{C}$ be a category. A *subobject classifier* in $\mathbf{C}$ is an object Ω and a (global) element $true\colon 1 \to \Omega$, such that for any monic arrow $s\colon S \rightarrowtail A$ there is a unique arrow $\chi_s\colon A \to \Omega$ such that the following diagram is a pullback:



The object Ω is called a *truth value object* and the unique arrow χ_s for each subobject s is called the *classifying arrow* or *characteristic arrow.* The arrow *true* is sometimes denoted as $\top$.

The subobject classifier, like the categorical structures mentioned in the previous section, is unique up to isomorphism, in the sense that the characteristic arrows between them are iso. Recall that one subobject $s\colon A \rightarrowtail C$ of C is contained in another $t\colon B \rightarrowtail C$ if there is an arrow from A to B that produces a commuting triangle. In such a case we write $s \subseteq t$. Also recall that two subobjects s and t are isomorphic if they contain each other, *i.e.* $s \subseteq t$ and $t \subseteq s$. In such a case we write $s \simeq t$. Now, it is not too hard to prove the following theorem:

[2] I abbreviate $true \circ !_Y$ as $true_Y$, which expresses that all of Y is mapped to 1, *i.e.* $true_Y$ is the function from Y to 2 that 'factors through' *true*.

[3] The commutativity follows becasue the terminal set 1 *separates mappings*, *i.e.* any two mappings with the same domain and codomain that agree on all the elements of the domain are identical.

Theorem 31 Let $s\colon A \rightarrowtail C$ and $t\colon B \rightarrowtail C$ be subobjects of C. Then

$$s \simeq t \Longleftrightarrow \chi_s = \chi_t.$$

Thus, characteristic arrows $\chi_{(\cdot)}\colon C \to \Omega$ are in 1-1 correspondence with equivalence classes of subobjects—each arrow from C to Ω picks out an equivalence class of subobjects. You will notice that the same is true in the category of abstract sets and arbitrary maps, since any two equinumerous subsets of a set are isomorphic.

Since toposes contain exponentials, it follows that given any object A, the object Ω^A exists. This is seen to be the object of all arrows from A to Ω, each of which, as we have just seen, is the the characteristic arrow of an equivalence class of subobjects of A. Thus, the object Ω^A picks out all of the subobjects of the object A. Since this is a generalization of the power set, Ω^A is called a *power object* of a topos. Power objects have a definition in terms of arrows, but it is complicated it will not be useful to consider it here.

An expanded definition of a topos is a category that has:

- all finite limits,
- all finite colimits,
- exponentials, and
- a subobject classifier.

Since a cartesian closed category is one that has all finite limits and exponentials, we see from this that cartesian closedness, together with the existence of a subobject classifier, ensures the existence of finite colimits. Thus, expanding our original definition, a topos is a category that has:

- all finite limits,
- exponentials, and
- a subobject classifier.

It is also possible to define an topos in terms of the existence of a power object instead of a subobject classifier. This equivalent definition of a topos is a category that has:

- all finite limits; and
- a power object.

Thus, the existence of a power object together with all finite limits is enough to ensure the existence of all finite colimits and exponentials for each pair of objects. Topos theory can be developed from this definition, which is desirable from a mathematical point of view because it is the most compact definition. Despite its advantages, Goldblatt (2006) has the following to say about this definition:

> Paedagogically it is not however the best [definition], for a number of reasons. Historically the idea of an elementary topos arose through the examination of subobject classifiers, and this path provides the most suitable motivation. As will be evident it is the Ω-axiom[, *i.e.* the existence of a subobject clasifier,] that is the key to the basic structure of a topos and it would have to be introduced anyway for the theory to get off the ground. Moreover, each of the Ω-axiom and the notion of exponentiation, is conceptually simpler than the description of power objects. (p. 106)

3 Examples of Toposes

With the definition of a topos in hand we may now consider some examples. The principal example of a topos, and one of the important motivations for the notion of a topos in the first place, is the category **Set** of sets and functions. The truth value object is, of course, the set $2 = \{0, 1\}$ and the subobject classifier is a map $\top\colon 1 \to 2$. The subcategory **FinSet** of finite sets and maps is also a topos, with the same classifying arrow as **Set**. The subcategory **Finord** of finite ordinals is also an example of a topos, since each finite set is isomorphic to a finite ordinal, so all the categorical constructions with finite sets are possible with finite ordinals. Also, the terminal object and the truth value object in all these cases are the finite ordinals $\{0\}$ and $\{0, 1\}$.

The category $\mathbf{Set}^2$, where $2 = \{0, 1\}$, of pairs $\langle X, Y\rangle$ of sets and pairs $\langle f, g\rangle$ of functions is another example of a topos. This really is a functor category since 2 is a category with two objects and only two arrows, the identity arrows. Where $1 = \{0\}$ is a terminal object in **Set**, a terminal object in $\mathbf{Set}^2$ is the set $\langle 1, 1\rangle$ and a subobject classifier is a function $\top_2\colon \langle 1, 1\rangle \to \langle 2, 2\rangle$, which can be thought of as an ordered pair $\langle \top, \top\rangle$ of subobject classifiers from **Set**. The categorical structures in this category are formed by "doubling up" the corresponding structures in **Set**. For example, given two arrows $\langle f, g\rangle\colon \langle A, B\rangle \to \langle E, F\rangle$ and $\langle h, k\rangle\colon \langle C, D\rangle \to \langle E, F\rangle$, if they form pullbacks

$$\begin{array}{ccc} P & \xrightarrow{j} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow[f]{} & E \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{v} & D \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[g]{} & F \end{array}$$

in **Set**, then

$$\begin{array}{ccc} \langle P, Q\rangle & \xrightarrow{\langle j,v\rangle} & \langle C, D\rangle \\ {\scriptstyle \langle i,u\rangle}\downarrow & & \downarrow{\scriptstyle \langle h,k\rangle} \\ \langle A, B\rangle & \xrightarrow[\langle f,g\rangle]{} & \langle E, F\rangle \end{array}$$

is a pullback in $\mathbf{Set}^2$.

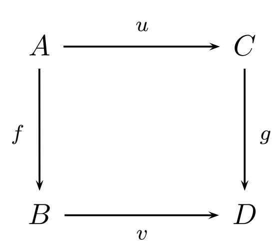
This situation generalizes. If I is any index set, then $\mathbf{Set}^I$ is also a topos. Objects of this topos are elements of $\prod_{i\in I} X_i$ for some sets X_i.

The topos $\mathbf{Set}^2$ category *could* be thought of as the simplest category of "varying" sets, it being the category of sets varying over 2. Similarly, given a set I, a category with card(I) objects and arrows, the topos $\mathbf{Set}^I$ would be the topos of sets varying over the index set I. This is not really a true kind of *variation*, however, since it is more like concatenation. Each of the elements of the index set I is unrelated to the others—the set I is a *discrete* category, *i.e.* the only arrows are identity arrows. So the objects of $\mathbf{Set}^I$ are just concatenations of a particular number, *viz.* card(I), sets. For a proper kind of variation, the domain category $\mathbf{D}$ of the functor category $\mathbf{Set}^{\mathbf{D}}$ must have more *structure*, which is to say that the category $\mathbf{D}$ must have non-identity arrows.

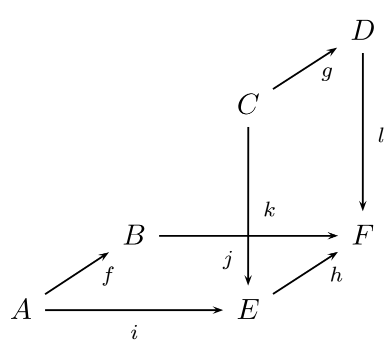
The simplest example of a topos that involves true *variation* is the category of "two-stage" sets, $\mathbf{Set}^{\rightarrow}$, or $\mathbf{Set}^{\mathbf{2}}$, where $\mathbf{2}$ is the category with two objects and three arrows:

In this case we have a category of sets varying over a "time-ordered" pair of objects. This "time-ordering," which is given by the arrow from 0 to 1, gives a definite sense to the objects of the topos being *variable* sets. You will notice that $\mathbf{Set}^{\rightarrow}$ is a category of *functions*, not simply a concatenation of sets. Since $\mathbf{Set}^{\rightarrow}$ is a functor category, 0 and 1 get mapped to some sets A and B respectively, and the arrow from 0 to 1 gets mapped to some function $f\colon A \to B$. The functor could be thought of as a "filling in" of the schema defined by $\mathbf{2}$.

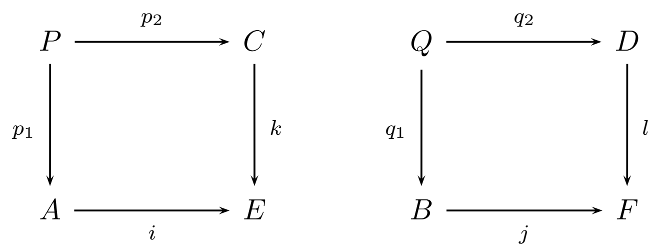
To get a sense of what the categorical structures in this category are like, first notice that an arrow in $\mathbf{Set}^{\rightarrow}$ between two objects $f\colon A \to B$ and $g\colon C \to D$ is a commutative square



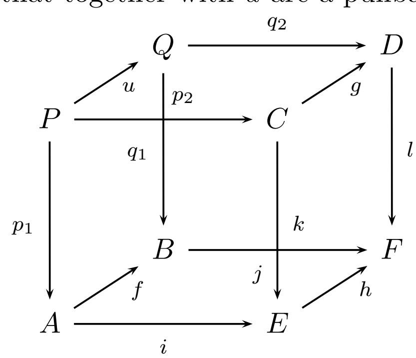
f is a subobject of g if both u and v are monic arrows. Given a "corner of arrows" $f \to h$ and $g \to h$ (i and j map f to h and k and l map g to h), where $f\colon A \to B$, $g\colon C \to D$, $h\colon E \to F$,



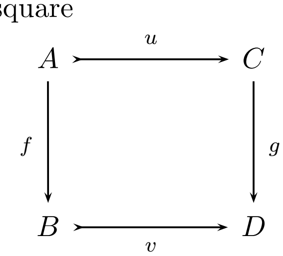
and pullbacks



in **Set**, there is a $u\colon P \to Q$ making the following diagram a pullback (the pair p_1 and q_1 of **Set**-arrows mapping u to f and the pair p_2 and q_2 of **Set**-arrows mapping u to g form the projection arrows in $\mathbf{Set^2}$ that together with u are a pullback in $\mathbf{Set^2}$):



The truth value object Ω in $\mathbf{Set}^{\to}$ is interesting. Let $f\colon\ A \to B$ be a subobject of $g\colon\ C \to D$, then there is a commutative **Set** square



We may interpret this as saying $A \subseteq C$ and $B \subseteq D$, and that f is the restriction of g to A. Now, given an element x of C, there are three ways of classifying it:

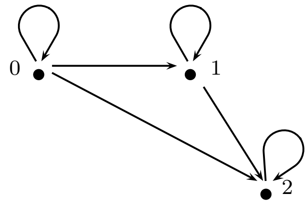
1. $x \in A$ (in which case $g(x) \in B$);
2. $x \notin A$, and $g(x) \notin B$; and
3. $x \notin A$, but $g(x) \in B$.

Thinking of f as a varying set contained in another varying set g, corresponding to the above three possibilities, there are three possibilities for "membership in f" of a (global) element x of g:

1. x is in f, in which case it is always in f (truth value 1);
2. x is never in f (truth value 0); and
3. x is "initially" not in f, *i.e.* $x \notin f$ at stage 0, but "eventually" is in f, *i.e.* $x \in f$ at stage 1.

Consequently, there are *three* truth values, calling the truth value for the "transitionary" case $\frac{1}{2}$, the truth value object is $\Omega = \left\{0, \frac{1}{2}, 1\right\}$. It is interesting to note that in the case of this topos the lattice of parts of Ω (ordered by inclusion) is *not* a boolean algebra, unlike the lattice formed from $\mathcal{P}(2)$ in **Set** which is, so the internal logic of this topos is intuitionistic.

The natural extension of the category $\mathbf{Set^2}$, is the topos $\mathbf{Set^3}$ of three-stage sets, where there are 4 truth values (always in, always out, out-in-in, and out-out-in). Then, for any natural number category $\mathbf{n}$, $\mathbf{Set^n}$ is a topos of n-stage sets. In the case of the natural number category $\mathbf{N}$, the category $\mathbf{Set^N}$ is also a topos, which could be seen as a category of sets with infinitely many discretely spaced stages.



That there are so many examples of toposes that are functor categories is a result of the fact that for any category $\mathbf{C}$, the functor category $\mathbf{Set^C}$ is a topos (for a proof see Bell (2008)). The ability to do this with any category gives rise to a vast range of possibilities for variable sets.

Consider the case of the natural numbers and real number as comprehended in set theory. Although the sets of natural numbers $\mathbb{N}$ and real numbers $\mathbb{R}$ thought of as categories are *discrete* categories, *i.e.* the only arrows are identity arrows, it can be useful to think of the toposes $\mathbf{Set}^{\mathbb{N}}$ and $\mathbf{Set}^{\mathbb{R}}$ as categories of variable sets. That $\mathbb{N}$ and $\mathbb{R}$ are discrete categories means that the toposes $\mathbf{Set}^{\mathbb{N}}$ and $\mathbf{Set}^{\mathbb{R}}$ do not involve variation in the strong sense of the examples just considered. But it can be useful to think of the objects of these toposes as 'time indexed' sets, discrete time indexing in the case of $\mathbf{Set}^{\mathbb{N}}$ and continuous time indexing in the case of $\mathbf{Set}^{\mathbb{R}}$ (where we appeal to the natural ordering of $\mathbb{N}$ and $\mathbb{R}$ to keep track of the ordering even though they are being regarded as discrete categories). In a similar way, the objects of the topos $\mathbf{Set}^{\mathbb{R}^3}$ could be thought of as 'space indexed' sets, a set being associated with each point in $\mathbb{R}^3$.

Another important class of toposes that are functor categories is comprised by categories of *monoid actions*. A monoid is a kind of algebraic structure, namely a set with an associative binary operation and an identity element. But it is not too hard to see that a monoid is a category with a single object—the object corresponds to the monoid itself, the arrows correspond to the elements of the monoid, the identity arrow corresponds to the identity element and composition of arrows gives the binary operation. Thus, given any monoid $\mathbf{M}$ thought of as a one object category, the functor category $\mathbf{Set^M}$ is a topos.

Let us take a look at the structure of the objects of this topos. Since each element of the monoid $\mathbf{M}$ is an arrow, each object of $\mathbf{Set^M}$ is a functor that maps an arrow of $\mathbf{M}$ to a function between two sets in $\mathbf{Set}$. But since $\mathbf{M}$ has only one object M, for any given functor $\boldsymbol{F}$ of $\mathbf{Set^M}$, the object M gets mapped to some set X and an arrow $a\colon M \to M$ gets mapped to a function $\boldsymbol{F}a\colon X \to X$ from X to itself. Thus, the images of the functors are collections of endomappings of some set. Now, given two arrows $a, b\colon M \to M$, that the mapping from $\mathbf{M}$ into $\mathbf{Set}$ is functorial entails that $\boldsymbol{F}(a \circ b) = \boldsymbol{F}a \circ \boldsymbol{F}b$. Thus, the function assigned to the monoid "product" $a \circ b$ of the arrows a and b of $\mathbf{M}$, is the same function obtained as the composite of the functions assigned to a and b individually. Thus, that the mapping from $\mathbf{M}$ into $\mathbf{Set}$ is functorial entails that the mapping from $\mathbf{M}$ to the collection of endomappings of X "respects the monoid structure." Such a mapping from a monoid to a collection of endomappings of a set is called a *monoid action.* A set X together with a collection of endomappings compatible with the structure of a monoid $\mathbf{M}$ is called an $\mathbf{M}$-*set.* Thus, given any monoid $\mathbf{M}$, the category of $\mathbf{M}$-sets is a topos. This category may also be denoted $\mathbf{M}$-$\mathbf{Set}$.

It might be helpful to think of a monoid action in terms of 'time evolution.' An element of the monoid could be thought of as a time shift, and the action of an element of the monoid on a set evolves that set forward in time. Elements of the set X get mapped to different elements, *i.e.* they move around in X over time. Thus, $\mathbf{M}$-sets are sets varying in time. A special case of a monoid action is a group action, since a group is a special case of a monoid. Recall that a group is a monoid that has inverses for all its elements. Reflecting the 'time evolution' analogy into the context of groups, group actions could be thought of as shifting sets through a space, and then $\mathbf{G}$-sets can be thought of as sets change as you shift locations in a space. This is because motions in time are not invertible, but motions in space are.

We can also consider functor categories $\mathbf{Set}^{\mathbf{C}}$ where the category $\mathbf{C}$ is some other kind of mathematical structure. Consider the case of a topological space X. What is important for the structure of a topological space are its collection of open sets and the inclusion relations between them. The collection of open sets related under inclusion forms a category, where the objects are the open sets U, V, *etc.* and the arrows are the inclusion maps $i_{U,V}\colon V \to U$, where $V \subset U$. The category of open sets of a topological space X can be denoted $\mathcal{O}_X$. Thus, the functor category $\mathbf{Set}^{\mathcal{O}_X}$ is a topos. This can be thought of as a category of sets that vary over a topological space.

The functor category of interest in this context is not exactly $\mathbf{Set}^{\mathcal{O}_X}$, really it is the category $\mathbf{Set}^{\mathcal{O}_X^{op}}$, where $\mathcal{O}_X^{op}$ is the *opposite category* of $\mathcal{O}_X$, *i.e.* the category obtained from $\mathcal{O}_X$ by reversing the arrows. The reason we are interested in functors of $\mathbf{Set}^{\mathcal{O}_X^{op}}$ is the following. Given a functor $\boldsymbol{F}$ in this category, it associates a set $\boldsymbol{F}U$ of data to each open set U of X and to each inclusion arrow $i_{U,V}\colon V \to U$ a *restriction function* $r_{U,V}\colon \boldsymbol{F}U \to \boldsymbol{F}V$ (notice that the order of U and V is reversed). It is the restriction functions we are interested in in mathematical contexts, not inclusion functions $\boldsymbol{F}V \to \boldsymbol{F}U$, since we often want to consider how some data defined on an open set behaves on parts of that open set. We can keep the notation $\mathbf{Set}^{\mathcal{O}_X}$ provided that it is understood that the functors are *contravariant* functors, *i.e.* functors that 'reverse' the direction of the arrows (for $f\colon A \to B$, $\boldsymbol{F}f\colon \boldsymbol{F}B \to \boldsymbol{F}A$).

The objects of the contravariant functor category $\mathbf{Set}^{\mathcal{O}_X}$ are called *presheaves.* The images $\boldsymbol{F}U$ of the open sets U of $\mathcal{O}_X$ are called *stalks* (of $\boldsymbol{F}$ over U) and their elements are called *sections* (of $\boldsymbol{F}$ over U) or *germs.* This is not a fully general definition of a presheaf, however, since a *presheaf* is any category of contravariant functors $\mathbf{C}^{\mathcal{O}_X}$, where $\mathbf{C}$ is any category. The category $\mathbf{Set}^{\mathcal{O}_X}$, then, is a category of presheaves of *sets.* We could also consider presheaves of groups, presheaves of rings, *etc.*

An important subclass of these categories presheaves are categories of *sheaves.* Sheaves are presheaves where the 'local behaviour determines the global behaviour' in a sense that I will make clear. We will just define sheaves in the case of a *concrete category* $\mathbf{C}$, *i.e.* a category such that each object A can be associated with its 'underlying set.' If s is a section in $\boldsymbol{F}U$, and $V \subset U$, then $s|_V$ is the *restriction* of s to V, which is the image of s under the restriction function $r_{U,V}$. In the case of a concrete category $\mathbf{C}$ a *sheaf* is a presheaf that satisfies the following axioms:

1. (Normalization) $\boldsymbol{F}\varnothing$ is the terminal object of $\mathbf{C}$;

2. (Local Identity) If $\{U_i\}$ is an *open covering* of an open set U, *i.e.* a set of open sets such that $U = \bigcup\{U_i\}$, and $s, t \in \boldsymbol{F}U$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i of the covering, then $s = t$;

3. (Gluing) If $\{U_i\}$ is an open covering of an open set U, and for each i, j there are sections s_i and s_j of $\boldsymbol{F}$ over U_i and U_j respectively that 'agree on the overlap' of U_i and U_j, *i.e.*
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$
then there is a section s of $\boldsymbol{F}$ over U that restricts to the s_i over U_i for each U_i of the covering, *i.e.* $s|_{U_i} = s_i$ for all i.

The section s that is guaranteed to exist by the third axiom is called the *gluing*, *concatenation* or *collation* of the sections s_i, which is unique by the second axiom. Sections that satisfy the conditions of the third axiom are called *compatible*. Thus, the second and third axioms together entail that *compatible sections can be glued together in a unique way.* This is the sense in which in a sheaf local behaviour determines the global behaviour. If the local behaviour of a s section over U, *i.e.* how it behaves on open subsets of U, is known, then the global behaviour of the section on the entirety of U is determined.

That what is important for the structure of a topological space X are its collection of open sets and the inclusion maps relating them offers an interesting way of abstracting away from concrete topological spaces. The ordering of objects determined by the arrows of the category $\mathcal{O}_X$, *i.e.* the inclusion maps, form a *lattice*, *i.e.* a partially ordered set such that any pair of elements has a supremum (meet) and an infimum (join), with an important special property. Every infinite collection $\{U_i\}$ of open sets of a topological space has a supremum $\bigcup\{U_i\}$ such that for any open set V of X
$$V \cap \bigcup\{U_i\} = \bigcup\{U_i \cap V\}.$$
In lattice theoretic terms this means that finite meets distribute over arbitrary joins. The structure of the space X is specified by the structure of the lattice $\mathcal{O}_X$. The *points* of each open set U, *i.e.* the elements of U, are immaterial to the structure of the space.

Just considering these lattices, and forgetting about the points, leads naturally to the concept of a *pointless space*. To define these we must first define a frame. A *frame* is a lattice such that finite meets distribute over arbitrary joins. The collection of frames together with lattice homomorphisms that respect arbitrary joins form a category, the category of frames. The opposite category **Loc** is the category of *pointless spaces*, or *locales*. The arrows of the category **Loc** of pointless spaces correspond to continuous functions between topological spaces. Thus, we see the manner in which this category generalizes the category **Esp** of topological spaces. The opposite category of the category of frames is considered as a result of the definition of a continuous map between topological spaces. A *continuous map* $f\colon X \to Y$ from a topological space X to another Y is a function from X to Y such that the inverse image of each open set in Y is an open set of X, *i.e.* for any open set U of Y, $f^{-1}(U)$ is open. Thus, a continuous map $f\colon X \to Y$ induces a map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. The sense of this induced map is opposite to that of the continuous function that induces it.

A lattice itself can be seen to be a category. The ordering of the lattice determines the arrows between its elements. Given a lattice L and elements $a, b \in L$, then there is an arrow from a to b just in case that $a \leq b$. It is not hard to see that the elements of the lattice together with these arrows satisfies the axioms for a category. Thus, given any lattice $\mathbf{Set}^L$ is a topos! And, thus, given any pointless space $\mathcal{S}$, $\mathbf{Set}^{\mathcal{S}}$ is the topos of sets varying over the pointless space.

Lattices are not the only kinds of ordered sets that form categories. A partially ordered set is also a category in the same way as a lattice. In fact, not even all the properties of a partial order are needed to satisfy the axioms for a category. For a set X together with a binary operation R to form a category the relation R must be *reflexive*, *i.e.* aRa for all $a \in X$, in order to have identity arrows, and the relation must be *transitive*, *i.e.* if aRb and bRc then aRc, in order to have composites of arrows. A binary relation that is reflexive and transitive is called a *preorder*. Thus, any preordered set P can be thought of as a category. Thus, given any preordered set P, $\mathbf{Set}^P$ is a topos, the topos of sets varying over the preordered set. The objects of the preordered set P can be thought of as different stages, states or positions in a network of relations, and the objects of $\mathbf{Set}^P$ are the sets varying over this network. In the way that the elements of a partially ordered set are thought of as 'states of knowledge' in the context of Kripke models, the topos of sets varying over a partially ordered set can be thought of as a category of sets varying over states of knowledge.

4 The Topos of (Constant) Sets

We have now seen a number of examples of toposes that can be thought of as categories of variable sets. Although we have not considered features of the structure of any of these toposes, many of them have properties quite unlike the category **Set** of sets and functions. This section is a consideration of the properties of **Set** that are considered to be characteristic of it as well as the properties of toposes that are considered to be **Set**-like. Before considering this we need to introduce some terminology and some basic facts about toposes.

One basic property that any topos must have if it is to be thought of as **Set**-like is that the initial object 0 should be like the empty set $\varnothing$ of **Set**. If this property is not satisfied then there is an arrow $x\colon 1 \to 0$ from 1 to 0. This has the catastrophic effect of making every object in the topos isomorphic. Such a topos is called *degenerate*. Thus, the initial object of every non-degenerate topos has no elements, and we rule out degenerate toposes as being **Set**-like.

Another basic property that **Set** has is that any object not isomorphic to the empty set $\varnothing$ is non-empty, *i.e.* contains elements. Thus, for a topos to be **Set**-like any object A not isomorphic to the initial object 0 must contain some element $x\colon 1 \to A$. In such a case we say that every *non-zero* object is *non-empty*. It is not true in every topos that each non-zero object is non-empty. For example, in $\mathbf{Set}^2$ $\langle\{0\}, \varnothing\rangle$ is non-zero, *i.e.* not isomorphic to $\langle\varnothing, \varnothing\rangle$, but it is empty since there are no $\mathbf{Set}^2$ arrows to it from $1 = \langle\{0\}, \{0\}\rangle$. Thus, for a topos to be **Set**-like, non-zero objects ought to be non-empty.

The category **Set** of sets has the property that sets are identified by their elements, *i.e.* any two sets that have the same elements are identical. This is called the *principle of* *extensionality*, or *extensionality* for short. Expressed categorically, *i.e.* in terms of arrows, this principle amounts to saying that the terminal object 1 *separates arrows*.

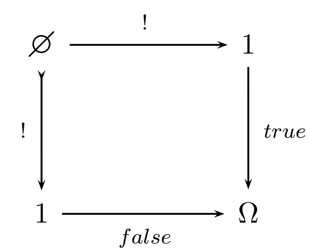
Definition 32 The terminal object 1 *separates arrows*, or simply 1 is a *separator*, if for any two distinct parallel arrows $f, g\colon A \to B$ there is an element $x\colon 1 \to A$ such that $f \circ x \neq g \circ x$.

This can be summarized as saying that any two distinct parallel arrows disagree on some (global) element of the domain. The condition that 1 is a separator is also called the *extensionality principle for arrows*. Not all toposes satisfy extensionality. For example, in $\mathbf{Set}^2$, there are two distinct arrows from $\langle \{0\}, \varnothing \rangle$ to $\langle 2, \varnothing \rangle$, but $\langle \{0\}, \varnothing \rangle$ has no elements to separate them. For a topos to be **Set**-like it must satisfy extensionality.

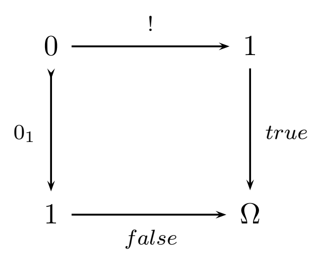
A non-degenerate topos that satisfies the extensionality principle for arrows is called *well-pointed*. Thus, any **Set**-like topos ought to be well-pointed. The well-pointedness condition implies a number of other properties. First of all, if a topos is well-pointed, then each non-zero object is non-empty. To discuss the other properties entailed by well-pointedness, it is necessary to first define the arrow $false$. In **Set** the truth value object is $2 = \{0, 1\}$. The the two elements $1 \to 2$ are the two truth values. $true$ is the arrow to 1 and the other, the arrow to 0, is the arrow $false$. Since this is an arrow to the truth value object, it must classify some subset. Indeed, the set it classifies is the set

$$\{x \mid false(0) = 1\} = \varnothing.$$

Thus, false classifies the empty set. This gives rise to the following commutative diagram, which is a pullback:



Generalizing this to an arbitrary topos, the arrow $false$ is defined to be the arrow such that the following diagram is a pullback:



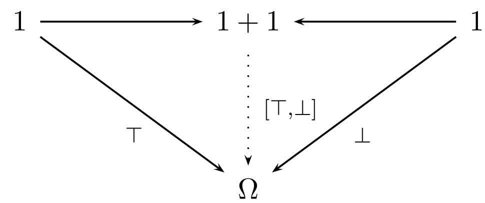
The arrow $false$ is also denoted by $\perp$. Thus, $false = \chi_{0_1}$. In any non-degenerate topos we are guaranteed that $true \neq false$. You might think of this situation intuitively as being such that the above diagram is trivially a pullback because $true$ and $false$ cover distinct, *i.e.* non-overlapping, parts of Ω. So the pullback of one along the other is the empty part $0 \rightarrowtail 1$ of 1.

An important property of the set of truth values in **Set** is that it contains only two elements, there are only two truth values. Thus, for a topos to be **Set**-like, then its truth value object ought to contain only two truth values.

Definition 33 A topos is called *bivalent* if *true* and *false* are its only truth values, *i.e.* elements of Ω.

It can be shown that if a topos is well-pointed then it is bivalent. Thus, well-pointedness also guarantees bivalency.

Another property of **Set** is that the coproduct $1+1$ is a two element set and hence isomorphic to $\Omega = 2$. The isomorphism is given by the unique arrow from the coproduct $1+1$ to Ω:



Since any topos has a terminal element and coproducts, the arrow $[\top, \bot]$ is always defined.

Definition 34 A topos in which $[\top, \bot]$ is an iso arrow is called *classical*.

We will see in the next section the manner in which any topos has an *internal* logic that is in general intuitionistic. The reason for this definition is that a classical topos is precisely one in which this internal logic is classical.

It can be shown that in any topos $[\top, \bot]$ is monic, but it is not true for all toposes that it is iso. Thus there are non-classical toposes. $\mathbf{Set}^{\rightarrow}$ is an example ($[\top, \bot]$ cannot be epic since Ω has three truth values). The truth value object of $\mathbf{Set}^{\rightarrow}$ actually consists of three functions from $2 = \{0, 1\}$ to 2. The truth value 1: $2 \rightarrow 2$ is defined by $1 \mapsto 1$ (in-in); the truth value 0: $2 \rightarrow 2$ is defined by $0 \mapsto 0$ (out-out); and the truth value $\frac{1}{2}$: $2 \rightarrow 2$ is defined by $0 \mapsto 1$ (out-in). Thus, in $\Omega = \left\{0, \frac{1}{2}, 1\right\}$ the elements are functions. Since the terminal object is the function 1: $\{0\} \rightarrow \{0\}$ there are three $\mathbf{Set}^{\rightarrow}$ arrows to Ω. Thus, Ω really does have three elements and $\mathbf{Set}^{\rightarrow}$ is not classical. In a similar way you may see that the topos $\mathbf{Set}^{\mathbf{n}}$ is not classical for any $\mathbf{n}$.

Now, it can also be shown that if a topos is well-pointed, then it is also classical. In fact, it can also be shown that a classical topos such that every non-zero object is non-empty is well-pointed. Thus, we have the following characterization of well-pointed toposes:

Theorem 35 A topos is well-pointed iff it is classical and every non-zero object is non-empty.

Another property of well-pointed toposes is that an arrow is surjective iff it is epic and injective iff it is monic (with the natural definition of injective and surjective in terms of arrows).

Another property that well-pointed toposes have is that they are *boolean*, a property that is actually equivalent to being classical. I will define this property in the following section. This property entails that any object A of the topos can be divided into two distinct (non-overlapping) parts a: $A_1 \rightarrowtail A$, $\overline{a}$: $A_2 \rightarrowtail A$ that can be recombined into the object A, *i.e.* their intersection is empty,

$$a \cap \overline{a} = 0_A,$$ [4]

[4]Notice the similarity to the law of non-contradiction.

where 0_A is the empty part of A, and when an appropriate definition of the union $f \cup g$ of two subobjects $f\colon B \rightarrowtail A$, $g\colon C \rightarrowtail A$ is defined,

$$a \cup \overline{a} = 1_A.$$[5]

Thinking of this in terms of the domains of the subobjects, this can be thought of as saying that $A_1 \cap A_2 = 0$ and $A_1 \cup A_2 = A$.

This kind of splitting of objects into two parts is not possible in all toposes. Toposes that have *cohesion* lack this property. For instance, in the topos **Spaces** of smooth spaces, which includes the smooth real line $\mathcal{R}$, the objects contain nilpotent infinitesimals. These (non-zero) infinitesimals act like 'glue' that holds together the points of $\mathcal{R}$.[6] If such objects are divided into non-overlapping parts then it is not possible to recombine them into the whole—in the process of division something is lost, the 'glue' that holds the parts together. To illustrate this idea Bell has drawn analogies to the breaking of a stick or the splitting of a piece of metal into parts have this property—once split into parts they cannot be restored to the whole by bringing them back together. Objects with cohesion cannot be reduced to their parts, or thought of in another way their parts are not independent of one another—they 'interact.' Thus, the parts of objects of toposes with the boolean property exhibit a kind of *independence* or *discreteness*.

A final important property characteristic of the category **Set** of sets is the *axiom of choice*. Categorically the axiom of choice can be stated in terms of the existence of a *section* for each epic arrow. Given a surjective function $f\colon X \to Y$ between two sets in **Set**, the *fibres* of the map are the inverse images $f^{-1}(y)$ of all the elements $y \in Y$. A section for such a function is a function $s\colon Y \to X$ such that $f \circ s = 1_Y$. Such a section can be thought of as a *choice function*. Given an element y of Y, Y being thought of as an index set, $s(y)$ is a choice of an element of the fibre $f^{-1}(y)$ over y (this is guaranteed by the definition $f \circ s = 1_Y$). Thus, a section of f selects an element from each fibre. It is then easy to see intuitively how the existence of a section for every surjective function is a form of the axiom of choice. The generalization to an arbitrary category is the following:

Axiom 1 (Epics Split (ES)) Each epic arrow $f\colon A \to B$ has a section $s\colon B \to A$ with $f \circ s = 1_B$.

There is a (strictly) stronger axiom of choice due to Mac Lane. This is the following:

Axiom 2 (Axiom of Choice (AC)) If $A \ncong 0$ then for any arrow $f\colon A \to B$ there exists a $g\colon B \to A$ such that $f \circ g \circ f = f$.

AC really is stronger than ES since it is possible to show that if AC holds in a topos $\mathcal{E}$, then ES holds in $\mathcal{E}$, and $\mathcal{E}$ is bivalent and has the property that non-zero objects are non-empty, but that ES on its own does not entail AC. A topos that satisfies ES and is well pointed, however, satisfies AC. It can also be shown that a topos satisfies AC iff it satisfies ES and every non-zero object is non-empty.

[5] Notice the similarity to the law of excluded middle.

[6] The points of $\mathcal{R}$ are the arrows $1 \to \mathcal{R}$, which does not exhaust the content of $\mathcal{R}$ as in the case of the set-theoretic reals $\mathbb{R}$. The set of infinitesimals of $\mathcal{R}$ is the set $\{x \mid \neg\neg x = 0\}$, 0 being the only point in the set of infinitesimals. Because the internal logic in **Spaces** is intuitionistic the set of infinitesimals contains more than just 0. The set of points of $\mathcal{R}$ is the set $\{x \mid \neg\neg\neg x = 0\}$.

The axiom of choice is a very strong principle. In fact, if the ES axiom of choice holds in a topos then that topos is boolean (classical), a result due to Diaconsecu. This imposes a kind of constancy as discussed above and it entails that the logic (internal and external) of the topos is classical. This is often summarized as saying that the axiom of choice implies the law of excluded middle.[7]

To sum up, we conclude that for a topos to be **Set**-like it must be a well-pointed topos. Well-pointed toposes actually are models of a weak form of Zermelo set theory.[8] Well-pointed toposes that also satisfy a property called *partial transitivity* are actually in exact correspondence with models of Zermelo set theory.[9] A characteristic of **Set** in particular, however, is that it also satisfies the axiom of choice. Well-pointed toposes that also satisfy ES are in exact correspondence with models of ZC (Zermelo set theory with choice).

There is another important property of **Set** that we have neglected to consider. **Set** contains a special infinite set ω, the least infinite ordinal. This is a result of the fact that **Set** satisfies the axiom of infinity. Categorically this amounts to the existence of a *natural numbers object*, which we will discuss below. That a category is well-pointed and satisfies ES is not sufficient to guarantee the existence of a natural numbers object. Thus, this must be added as an axiom in order to characterize **Set**. Thus, **Set** is a *well-pointed topos with a natural numbers object in which epics split*.[10]

5 Internal and External Aspects of Topos Logic

The connection of topos theory to logic is quite varied and complex. This section is devoted to a consideration of some of the basic aspects of topos logic. This will serve to clarify the status of the so-called *internal logic* of a topos. Using this internal logic, a topos admits an interpretation of a certain kind of higher order (intuitionistic) type theory. Such theories are called *local set theories*. In fact, each local set theory can be seen to generate a topos, which is called a *linguistic topos*, and it turns out that each topos is *equivalent* to a linguistic one.

To get started we will consider some *external* aspects of topos logic. Another property of **Set** not considered in the previous section is that given any set X, $\mathcal{P}(X)$ is a *boolean algebra* (a complemented distributive lattice) with the ordering given by set inclusion. Given any topos $\mathcal{E}$ and an object A, an analogue of this is $\mathrm{Sub}(A)$, which is the collection of equivalence

[7]The axiom of choice (ES or AC) also implies that a topos is *weakly extensional* or *localic*. This is the property that *parts* of 1 separate arrows. That a topos is well pointed, however, is not sufficient to make the axiom of choice hold, which is established by Cohen's work on independence proofs that showed that "there are models of set theory, hence well-pointed [toposes], in which the axiom of choice fails." (Goldblatt, 2006, 299-300)

[8]This form of set theory, denoted Z_0, includes the axioms of classical first-order logic with identity and the axioms of extensionality, empty set, pairing, powerset, union and bounded separation.

[9]This system, denoted Z, satisfies the axioms of Z_0 as well as regularity, the axiom of transitivity and the axiom of transitive representation (see Goldblatt (2006)).

[10]The category of sets is also well-founded, it does not have cyclic membership or infinite descending membership chains. This is a result of the fact that **Set** satisfies the axiom of regularity. The system Z_0 together with infinity, regularity and replacement (this ensures that the image of a function whose domain is a set is also a set) is the system ZF of Zermelo-Fraenkel set theory. The (apparently) appropriate framework for constructing categorical models of ZF is a more general framework than toposes. Joyal & Moerdijk (1995) work in a framework of heyting pretoposes and the models of (intuitionistic) ZF are what they call *free Zermelo-Fraenkel algebras*.

classes of subobjects of A and is isomorphic to the collection of arrows $\mathcal{E}(A, \Omega)$. Under the ordering of inclusion $\subseteq$ of subobjects, $\mathrm{Sub}(A)$ is a partially ordered set.[11] Given two subobjects $f\colon B \rightarrowtail A$ and $g\colon C \rightarrowtail A$ of A, it is possible to define their intersection $f \cap g$ (in terms of the pullback of f along g) and their union $f \cup g$ (in terms of the coproduct $B + C$ and epi-monic factorization). $f \cap g$ and $f \cup g$ are the infimum (meet) and supremum (join) of f and g, respectively. Thus, $\mathrm{Sub}(A)$ is a lattice. In fact, it is a bounded lattice with top element 1_A and bottom element 0_A. Moreover, it is distributive, since

$$f \cap (g \cup h) \cong (f \cup g) \cap (f \cup h).$$

The question, then, is whether $\mathrm{Sub}(A)$ is a complemented lattice, making it a boolean algebra. In fact, it is in general not.

Recall that a bounded lattice $\langle L, \leq \rangle$ with top and bottom elements 1 and 0 is *complemented* if any a in L has a *complement* a', *i.e.* there is an $a' \in L$ such that

$$a \cap a' = 0 \qquad \text{and} \qquad a \cup a' = 1.$$

In terms of a boolean algebra providing a semantics for classical logic, the former corresponds to the law of non-contradiction and the latter to the law of excluded middle. Both of these properties are not satisfied in $\mathrm{Sub}(A)$ for an arbitrary topos. A form of the former is. The definition of negation $\neg$ in a topos is the arrow making the following diagram is a pullback:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \bot}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ \Omega & \xrightarrow[\neg]{} & \Omega \end{array}$$

i.e. $\neg$ is the classifying arrow $\chi_\bot$ of $\bot$. Given a subobject $f\colon B \rightarrowtail A$, $-f$ is defined to be the subobject classified by $\chi_{-f} = \neg \circ \chi_f$. $-f$ is the correlate in an arbitrary topos of the (relative) complement of a subset in $\mathcal{P}(X)$ (*cf.* the discussion of boolean toposes in the previous section). It is true in an arbitrary topos that

$$f \cap -f = 0_A.$$

It is not, however, true in all toposes that

$$f \cup -f = 1_A.$$

Thus, $\mathrm{Sub}(A)$ does not in general provide a semantics for classical logic.

Definition 36 A topos in which $(\mathrm{Sub}(A), \subseteq)$ for any object A is a boolean algebra is called a *boolean* topos.

It can be shown that a topos is boolean iff it is classical. Thus, well-pointed toposes are boolean. It can also be shown that a topos is boolean iff $i\colon 1 \to 1+1$ is a subobject classifier and iff $\top \cup \bot = 1_\Omega$ in $\mathrm{Sub}(\Omega)$. It can also be shown that a topos is boolean iff $\mathrm{Sub}(\Omega)$ is a boolean algebra. Thus, $\mathrm{Sub}(\Omega)$ being a boolean algebra is sufficient to ensure that $\mathrm{Sub}(A)$ is a boolean algebra for all objects A.

[11] We assume here some ambient 'set' theory which gives meaning to 'collection.' This highlights that $\mathcal{E}(A, \Omega)$ and $\mathrm{Sub}(A)$ are *external* to the topos, *i.e.* not objects of the topos.

Toposes can provide a semantics for propositional logic in another way than in terms of the algebra Sub(A) of subobjects of an object A. To describe how this works it is necessary to define the logical connectives, negation $\neg$, conjunction $\cap$, disjunction $\cup$ and implication $\mapsto$ in terms of arrows. Consider first the situation in **Set**. In line with what we saw above, negation $\neg\colon 2 \to 2$ is the characteristic function of the set

$$\{x \mid \neg x = 1\} = \{0\} \subseteq 2.$$

Since the inclusion function $\{0\} \to 2$ is just the function $false$, we have the following pullback in **Set**:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle false}\downarrow & & \downarrow{\scriptstyle true} \\ 2 & \xrightarrow[\neg]{} & 2 \end{array}$$

This explains the definition of $\neg$ in an arbitrary topos above.

In the case of conjunction $\cap\colon 2 \times 2 \to 2$, the only input that gives output 1 is $\langle 1, 1\rangle$. Thus, $A = \{\langle 1, 1\rangle\}$ is the part of 2×2 that gets mapped to 1, so that $\cap = \chi_A$. Now, the pullback of $true$ along $\cap$ is the map $1 \to 2 \times 2$ that picks out A, *i.e.* the map defined by $\{0\} \mapsto \langle 1, 1\rangle$, thus the following is a pullback:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \langle true, true\rangle}\downarrow & & \downarrow{\scriptstyle true} \\ 2 \times 2 & \xrightarrow[\cap]{} & 2 \end{array}$$

Generalizing this to an arbitrary topos we have that conjunction $\cap$ is the arrow from $\Omega \times \Omega$ to Ω such that the following is a pullback:

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \langle \top, \top\rangle}\downarrow & & \downarrow{\scriptstyle \top} \\ \Omega \times \Omega & \xrightarrow[\cap]{} & \Omega \end{array}$$

The construction of $\cup$ and $\mapsto$ in terms of arrows is more complicated and it will not be useful to develop them here. Thus, I will just give the definitions. The interested reader can find the motivation for these definition in Goldblatt (2006, 137-9). Let $\mathcal{E}$ be a topos. The disjunction arrow $\cup\colon \Omega \times \Omega \to \Omega$ is defined to be the character of the image of the $\mathcal{E}$ arrow

$$[\langle \top, 1_\Omega\rangle, \langle 1_\Omega, \top\rangle]\colon \ \Omega + \Omega.$$

The implication arrow $\mapsto\colon \Omega \times \Omega \to \Omega$ is defined to be the character of the subobject

$$e\colon \ \preceq \rightarrowtail \Omega \times \Omega,$$

which is the equalizer of

$$\Omega \times \Omega \overset{\cap}{\rightrightarrows} \Omega,$$

where $\cap$ is the conjunction arrow and the lower arrow is p_1, the first projection arrow of the product $\Omega \times \Omega$.

Now we may consider how the truth values of a topos can provide the semantics for propositional logic. This is done in terms of a *truth valuation*, which is a mapping $V\colon \mathcal{L} \to \mathcal{E}(1, \Omega)$ from the language $\mathcal{L}$ of propositional logic to the collection of truth values $\mathcal{E}(1, \Omega)$. The truth valuation is defined in the following way. First atomic propositions p in the language $\mathcal{L}$ are assigned truth values, *i.e.* arrows $V(p)\colon 1 \to \Omega$. This is extended to a truth valuation on all of $\mathcal{L}$ using the negation $\neg$, conjunction $\cap$, disjunction $\cup$ and implication $\mapsto$ operators of $\mathcal{E}$, defined as arrows above, and defining

$$\begin{aligned} V(\neg p) &= \neg \circ V(p), \qquad (1)\\ V(p \wedge q) &= \cap \circ \langle V(p), V(q) \rangle \\ V(p \vee q) &= \cup \circ \langle V(p), V(q) \rangle \\ V(p \to q) &= \mapsto \circ \langle V(p), V(q) \rangle. \end{aligned}$$

If a given proposition p is assigned the truth value $\top$, then we write

$$\mathcal{E} \vDash p.$$

Now, it can be shown that if a topos $\mathcal{E}$ is boolean, then $\mathcal{E} \vDash p \vee \neg p$ for any sentence p. In fact it can be shown that for any topos $\mathcal{E}$, $\mathcal{E} \vDash p \vee \neg p$ iff $\mathrm{Sub}(1)$ is a boolean algebra. What is perhaps peculiar is that a topos can fail to be boolean yet it is still the case that $\mathcal{E} \vDash p \vee \neg p$. This is because there ere are toposes where $\mathrm{Sub}(1)$ is a boolean algebra but $\mathrm{Sub}(\Omega)$ is not. This situation is clarified with the observation, mentioned in a footnote above, that the collections $\mathrm{Sub}(A)$ and $\mathcal{E}(1, \Omega)$ are *not* objects in the topos! They are *external* to $\mathcal{E}$. The objects of $\mathcal{E}$ that correspond to these collections are Ω^A and $\Omega^1 \cong \Omega$ respectively. Thus, we are led to consider the *internal* version of the law of excluded middle.

In **Set** the validity of $p \vee \neg p$ corresponds to the equation $x \cup \neg x = 1$ for any $x \in 2$. In terms of commuting diagrams, this is equivalent to the commutativity of

$$\begin{array}{ccc} 2 & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle \langle 1_2, \neg \rangle} \downarrow & & \downarrow {\scriptstyle \mathit{true}} \\ 2 \times 2 & \xrightarrow[\cup]{} & 2 \end{array}$$

This is generalized to an arbitrary topos $\mathcal{E}$ as the commutativity of the following diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle \langle 1_\Omega, \neg \rangle} \downarrow & & \downarrow {\scriptstyle \top} \\ \Omega \times \Omega & \xrightarrow[\cup]{} & \Omega \end{array}$$

The commutativity of this diagram is equivalent to the internal validity of the law of excluded middle since it can be shown that in a topos $\mathcal{E}$ $\mathrm{Sub}(\Omega)$ is a boolean algebra iff this diagram commutes.

Now, a set of axioms for classical propositional logic is the following:

1. $p \to (p \wedge p)$
2. $(p \wedge q) \to (q \wedge p)$
3. $(p \to q) \to ((p \wedge r) \to (q \wedge r))$
4. $((p \to q) \wedge (q \to r)) \to (p \to r)$
5. $q \to (p \to q)$
6. $(p \wedge (p \to q)) \to q$
7. $p \to (p \vee q)$
8. $(p \vee q) \to (q \to p)$
9. $((p \to r) \wedge (q \to r)) \to ((p \vee p) \to r)$
10. $\neg p \to (p \to q)$
11. $((p \to q) \wedge (p \to \neg q)) \to \neg p$
12. $p \vee \neg p$

If the last axiom, the law of excluded middle is left out, then axioms 1-11 form a set of axioms for intuitionistic propositional logic. All of these can be written in the form of a commutative diagram. It can be shown that in any topos $\mathcal{E}$ the diagrams for axioms 1 to 11 all commute. Thus, the axioms of intuitionistic propositional logic are internally true in every topos. Axiom 12 is internally true in a topos iff it is boolean. Thus, the internal (propositional) logic of a topos is in general intuitionistic and the internal logic is classical iff the topos is boolean.

Up until now we have only considered propositional logical operators. The interpretation of the existential and universal quantifiers in toposes is (significantly!) more complicated and involves the notion of an *adjunction* or *adjoint situation*. This notion is developed briefly in appendix A. First note that a truth valuation of the sentences

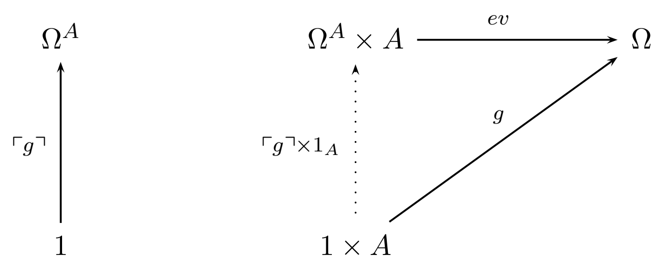
$$\forall x \varphi(x) \quad \text{and} \quad \exists x \varphi(x),$$

can be thought of in terms of the the universal and existential operators assigning truth values to properties. For example,

$$\forall x \varphi(x) = \mathit{true}$$

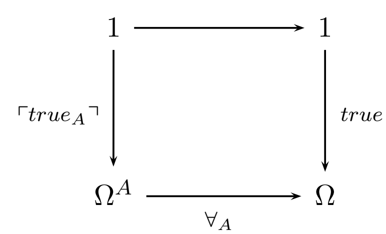
can be thought of as the universal quantifier $\forall$ assigning the property φ the truth value *true*. Recall that properties of an object A in a topos are arrows $A \to \Omega$ from A to the truth value object Ω. Thus, for properties φ_A of A, the assignment of a truth value to sentences $\forall_A x \varphi(x)$ and $\exists_A x \varphi(x)$ ought to send an element of Ω^A to an element of Ω. Thus, we would expect the internal interpretations of the quantifiers to be arrows $\Omega^A \to \Omega$.

The interpretation of the universal quantifier is simple enough to consider here. The existential quantifier is much more difficult to analyze and the interested reader should consult Goldblatt (2006, 245 *ff.*). To develop the definition we require the notion of the *name* of an arrow, which we just consider in the special case here. Consider the exponential object Ω^A and its evaluation arrow $ev\colon \Omega^A \times A \to \Omega$. From the definition of the exponential, given an arrow $1 \times A \to \Omega$, which corresponds to an arrow $A \to \Omega$, since $A \cong 1 \times A$, we have the following commutative diagram:



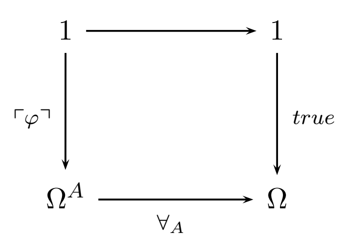
The arrow $\ulcorner g \urcorner$, the exponential adjoint of g, is called the *name* of the arrow g. It is the element of Ω^A corresponding to the arrow g. The general definition of the name of an arrow is similar.

Now, the intuitive idea for the interpretation of universal quantification is that if $\forall_A x \varphi(x)$ is true, then all parts of A receive the truth value *true*. Thus we will require the map $true_A\colon A \to \Omega$ that maps all of A onto *true*, *i.e.* the map that factors through $true\colon 1 \to \Omega$, and we need to connect this to the name of arrow $true_A$. The way to accomplish this is to consider the composite $true_A \circ p_1\colon 1 \times A \to A \to \Omega$, so that the name $\ulcorner true_A \urcorner$ of the arrow $true_A$ is the exponential adjoint of $true_A \circ p_1$. Then, the universal quantifier is the unique arrow $\forall_A\colon \Omega^A \to \Omega$ making the following diagram a pullback:



We see that the universal quantifier sends elements of Ω^A to elements of Ω as required.

Now, given a property $\varphi\colon A \to \Omega$, will analyze the truth of the proposition $\forall_A x\, \varphi(x)$ in terms of the name $\ulcorner \varphi \urcorner$ of the property.[12] If the proposition $\forall_A x\, \varphi(x)$ is true, then the following diagram commutes:



Thus, if we interpret the proposition $\forall_A x\, \varphi(x)$ as $\forall_A \circ \ulcorner \varphi \urcorner$, then the truth of the proposition,

[12] The name of a property φ is a special case of a *power transpose* of φ, which is introduced below.

and the commutativity of the above diagram, is expressed by

$$\forall_A \ulcorner\varphi\urcorner = true.$$

You will see, then, that

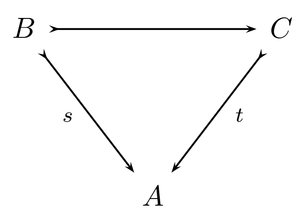
$$\forall_A \ulcorner\varphi\urcorner = true \Longleftrightarrow \ulcorner\varphi\urcorner = \ulcorner true_A \urcorner.$$

6 The Internal Logic of a Topos and Local Set Theories

Now that we have an appreciation of some of the basics of topos logic, we may now turn to consider a nice way of organizing and clarifying logic and set theory in a topos. This involves the notions of a *local language* and a *local set theory*. The basic idea of how a topos gives rise to a 'local' set theory is the following. The objects of the topos are thought of as *types* in the logical sense, but Bell (2005) suggests that these types may be thought of as *natural kinds* or *species*. Unlike in set theory, however, where the inclusion relation among sets is global, the inclusion relation among the types, *i.e.* objects, of a topos is only defined between subtypes or subspecies of the same type. This is because a subobject is an arrow

$$s \colon B \rightarrowtail A$$

and one subobject $s \colon B \rightarrowtail A$ is included in another $t \colon C \rightarrowtail A$, *i.e.* $s \subseteq t$, just in case that the following diagram commutes:



Thus, inclusion is only defined for two subspecies of the same species, *i.e.* A—inclusion is *local*. Such a theory where the objects in a topos are thought of as sets is in this sense a *local set theory*.

A local set theory is a higher order type theory with the same primitive symbols as classical set theory, *viz.* $=$, $\in$ and $\{\,|\,\}$, where set-theoretic operations of forming products and powers of types can be performed and that contains a truth value type, which acts as the range of values of propositional functions on types. (Bell, 2005, 297) A local set theory is formulated by specifying a set of 'extra logical' axioms within a local language. Thus, we first define a local language.[13]

A *local language* $\mathcal{L}$ contains the following symbols:

- $\mathbf{1}$ (unity type)
- Ω (truth value type)
- $\mathbf{S}, \mathbf{T}, \mathbf{U}, \ldots$ (ground types, possibly none)
- $\mathbf{f}, \mathbf{g}, \mathbf{h} \ldots$ (function symbols, possibly none)

[13]The exposition in this section follows Bell (2005) very closely.

- $x_{\mathbf{A}}, y_{\mathbf{A}}, z_{\mathbf{A}} \ldots$ (variables of each type $\mathbf{A}$, defined below)
- $\bullet$ (unique entity of type 1)

The *types* of $\mathcal{L}$ are defined recursively in the following way:

- $\mathbf{1}$ and Ω are types
- any ground type is a type
- $\mathbf{A_1} \times \cdots \times \mathbf{A_n}$ is a type whenever $\mathbf{A_1}, \ldots, \mathbf{A_n}$ are types and if $n = 1$ $\mathbf{A_1} \times \cdots \times \mathbf{A_n}$ is $\mathbf{A_1}$ and $\mathbf{1}$ if $n = 0$ (product types)
- $\mathbf{PA}$ is a type whenever $\mathbf{A}$ is (power types)

Each function symbol $\mathbf{f}$ has a *signature* of the form $\mathbf{A} \to \mathbf{B}$ for some types $\mathbf{A}$ and $\mathbf{B}$. The signature of a function symbol may be indicated as $\mathbf{f}\colon \mathbf{A} \to \mathbf{B}$.

The terms of a local language $\mathcal{L}$ are defined recursively in the following way, where $\tau\colon \mathbf{A}$ indicates that the term τ has type $\mathbf{A}$.

Term: **type**	Conditions
$\bullet : \mathbf{1}$	
$x_{\mathbf{A}}\colon \mathbf{A}$	
$\mathbf{f}(\tau)\colon \mathbf{B}$	$\mathbf{f}\colon \mathbf{A} \to \mathbf{B}$, $\tau\colon \mathbf{A}$
$\langle \tau_1, \ldots, \tau_n \rangle\colon \mathbf{A_1} \times \cdots \times \mathbf{A_n}$, $\langle \tau_1, \ldots, \tau_n \rangle$ is τ_1 if $n = 1$ and $\bullet$ if $n = 0$	$\tau_1\colon \mathbf{A_1}, \ldots, \tau_{\mathbf{n}}\colon \mathbf{A_n}$
$(\tau)_i\colon \mathbf{A_i}$, $(\tau)_i$ is τ if $n = 1$	$\tau\colon \mathbf{A_1} \times \cdots \times \mathbf{A_n}$, $1 \leq i \leq n$
$\{x_{\mathbf{A}} \mid \alpha\}\colon \mathbf{PA}$	$\alpha\colon \Omega$
$\sigma = \tau\colon \Omega$	σ and τ of same type
$\sigma \in \tau\colon \Omega$	$\sigma\colon \mathbf{A}, \tau\colon \mathbf{PA}$ for some type $\mathbf{A}$

Terms of type Ω are called *formulas*, *propositions* or *truth values*. Formulas will be denoted by Greek letters from the beginning of the alphabet, $\alpha, \beta, \gamma, \ldots$ and variables of type Ω will be denoted $\omega, \omega', \omega'', \ldots$. If it is clear what the type of a variable $x_{\mathbf{A}}$ is, we may drop the explicit reference to the type $\mathbf{A}$ and just write x. Given a term τ, we denote the result of the substitution of a term σ for each free occurrence of the variable x by $\tau(x/\sigma)$ (an occurrence of x is *free* if it does not occur in $\{x \mid \alpha\}$). We will also write $\alpha \leftrightarrow \beta$ if $\alpha = \beta$. A term is called *closed* if it has no free variables. A closed term of type Ω, *i.e.* a closed formula, is called a *sentence*.

Using the sequent notation, *i.e.* $\Gamma\colon \alpha$ for a finite set of sentences Γ, to express that α is a logical consequence of the sentences Γ, the *basic axioms* for a local language are the following:

Tautology	$\alpha\colon \alpha$
Unity	$:\ x_1 = \bullet$
Equality	$x = y, \alpha(z/x)\colon \alpha(z/y),\ x, y$ free for z in α
Products	$:\ (\langle x_1, \ldots, x_n \rangle)_i = x_i$
Comprehension	$x \in \{x \mid \alpha\} \leftrightarrow \alpha$

The *rules of inference* are the following:

Thinning $\dfrac{\Gamma\colon\ \alpha}{\beta, \Gamma\colon\ \alpha}$

Restricted Cut $\dfrac{\Gamma\colon\ \alpha \quad \alpha, \Gamma\colon\ \beta}{\Gamma\colon\ \beta}$ (where any free variable of α is free in Γ or β)

Substitution $\dfrac{\Gamma\colon\ \alpha}{\Gamma(x/\tau)\colon\ \alpha(x\tau)}$ (τ not free for x in Γ and α)

Extensionality $\dfrac{\Gamma\colon\ x \in \sigma \leftrightarrow x \in \tau}{\Gamma\colon\ \sigma = \tau}$ (x not free in Γ, σ, τ)

Equivalence $\dfrac{\alpha, \Gamma\colon\ \beta \quad \beta, \Gamma\colon\ \alpha}{\Gamma\colon\ \alpha \leftrightarrow \beta}$

The axioms and rules of inference for a local language give rise to a system of natural deduction in $\mathcal{L}$. If S is any collection of sequents in $\mathcal{L}$ then the sequent $\Gamma\colon\ \alpha$ is *deducible from* S, written $\Gamma \vdash_S \alpha$, if there is a deduction of $\Gamma\colon\ \alpha$ using the basic axioms, the sequents of S and the rules of inference. In case that $S = \varnothing$, we write $\Gamma \vdash \alpha$ and if $\Gamma = \varnothing$, we write $\vdash_S \alpha$.

A *local set theory* in a local language $\mathcal{L}$ is a collection S of sequents that is closed under deducibility from S. Given any collection of sequents S, this generates the local set theory S^*, the deductive closure of S. The local set theory in $\mathcal{L}$ generated by $\varnothing$ is called *pure* local set theory in $\mathcal{L}$.

Now, notice that there are no logical operators among the primitive symbols of a local language $\mathcal{L}$. The reason for this is that the logical operators can actually be *defined* in terms of the primitive symbols of $\mathcal{L}$—a local set theory generates its own *internal* logic. The *logical operations* in a local set theory are defined in the following way:

Logical Operation	Definition
$\alpha \leftrightarrow \beta$	$\alpha = \beta$
$\top$ (true)	$\bullet = \bullet$
$\alpha \wedge \beta$	$\langle \alpha, \beta \rangle = \langle \top, \top \rangle$
$\alpha \rightarrow \beta$	$(\alpha \wedge \beta) \leftrightarrow \alpha$
$\forall x\, \alpha$	$\{x \mid \alpha\} = \{x \mid \top\}$
$\bot$ (false)	$\forall \omega\, \omega$
$\neg \alpha$	$\alpha \rightarrow \bot$
$\alpha \vee \beta$	$\forall \omega[(\alpha \rightarrow \omega \wedge \beta \rightarrow \omega) \rightarrow \omega]$
$\exists x\, \alpha$	$\forall \omega[\forall x(\alpha \rightarrow \omega) \rightarrow \omega]$

As is customary, $x \neq y$ is defined to be $\neg(x = y)$, $x \notin y$ is defined to be $\neg(x \in y)$ and $\exists! x\, \alpha$ is defined to be $\exists x[\alpha \wedge \forall y\, \alpha(x/y) \rightarrow x = y]$.

Examples of theorems that are derivable in any local set theory are the introduction and elimination rules for the logical operators. In the case of conjunction we have that

$$\frac{\Gamma\colon\ \alpha \quad \Gamma\colon\ \beta}{\Gamma\colon\ \alpha \wedge \beta} \quad \text{and} \quad \frac{\Gamma\colon\ \alpha \wedge \beta}{\Gamma\colon\ \alpha \quad \Gamma\colon\ \beta}.$$

In the case of implication we have that

$$\frac{\alpha, \Gamma\colon\ \beta}{\Gamma\colon\ \alpha \rightarrow \beta} \quad \text{and} \quad \frac{\Gamma\colon\ \alpha \rightarrow \beta}{\alpha, \Gamma\colon\ \beta}.$$

We also have the following relation between equivalent propositions and their extensions:

$$\frac{\Gamma\colon\ \alpha \leftrightarrow \beta}{\Gamma\colon\ \{x \mid \alpha\} = \{x \mid \beta\}}.$$

For additional results and proofs see Bell (2008, 73-83).

With the logical operations defined, we may now consider how the concept of a set is introduced into a local language. A *set-like* term is a term of power type and a *closed* set-like term is called an ($\mathcal{L}$-)*set*. Sets are denoted by upper case roman letters X, Y, Z, we use $\forall x \in X\, \alpha$ to denote $\forall x(x \in X \leftrightarrow \alpha)$ and a similar abbreviation for the existential quantifier. The relations and operations for sets in a local language are the following (note that in the definitions of $\subseteq, \cap$ and $\cup$, the sets X and Y must be of the same type):

Operation	Definition
$\{x \in X \mid \alpha\}$	$\{x \mid x \in X \wedge \alpha\}$
$X \subseteq Y$	$\forall x \in X\, x \in Y$
$X \cap Y$	$\{x \mid x \in X \wedge x \in Y\}$
$X \cup Y$	$\{x \mid x \in X \vee x \in Y\}$
U_A or A	$\{x_{\mathbf{A}} \mid \top\}$
$\varnothing_A$ or $\varnothing$	$\{x_{\mathbf{A}} \mid \bot\}$
$E - X$	$\{x \mid x \in E \wedge x \notin X\}$
PX	$\{u \mid u \subseteq X\}$
$\bigcap U (U\colon\ \mathbf{PPA})$	$\{x \mid \forall u \in U\, x \in u\}$
$\bigcup U (U\colon\ \mathbf{PPA})$	$\{x \mid \exists u \in U\, x \in u\}$
$\bigcap_{i \in I} X_i$	$\{x \mid \forall i \in I\, x \in X_i\}$
$\bigcup_{i \in I} X_i$	$\{x \mid \exists i \in I\, x \in X_i\}$
$\{\tau_1, \ldots, \tau_n\}$	$\{x \mid x = \tau_1 \vee \cdots \vee x = \tau_n\}$
$\{\tau \mid \alpha\}$	$\{z \mid \exists x_1 \cdots \exists x_n (z = \tau \wedge \alpha)\}$
$X \times Y$	$\{\langle x, y\rangle \mid x \in X \wedge x \in Y\}$
$X + Y$	$\{\langle \{x\}, \varnothing\rangle \mid x \in X\} \cup \{\langle \varnothing, \{y\}\rangle \mid y \in X\}$
$Fun(X, Y)$	$\{u \mid u \subseteq X \times Y \wedge \forall x \in X \exists! y \in Y\, \langle x, y\rangle \in u\}$

The important fact about the collection of $\mathcal{L}$-sets of a local set theory is that is determines a topos. Let S be a local set theory in a local language $\mathcal{L}$. We define an equivalence relation $\sim_S$ on the collection of all $\mathcal{L}$-sets by

$$X \sim_S Y \Longleftrightarrow \vdash_S X = Y.$$

An S-*set* is then defined to be a equivalence class $[X]$ of $\mathcal{L}$-sets under this equivalence relation, which we may identify with X. An S-*map* $f\colon\ X \rightarrow Y$ is a triple (f, X, Y), which may be identified with f, of S-sets such that $\vdash_S f \in Y^X$. X and Y are the *domain* and *codomain*, respectively, of the S-map. The collection of S-sets and S-maps forms a category, with the composite of S-maps $f\colon X \to Y$ and $g\colon Y \to Z$ being given by

$$g \circ f = \{\langle x, z\rangle \mid \exists y(\langle x, y\rangle \in f \wedge \langle y, z\rangle \in g)\}.$$

Moreover, this category is a *topos*, the *topos of sets determined by S*.

We are now ready to see how a local language is interpreted in a topos, *i.e.* how a topos can provide the semantics for a local language. Let $\mathcal{L}$ be a local language and $\mathcal{E}$ a topos. Then, an *interpretation I* of $\mathcal{L}$ in $\mathcal{E}$ is the following kind of assignment:

- each type $\mathbf{A}$ is assigned an $\mathcal{E}$-object A_I such that the following conditions are met:
    1. $(\mathbf{A}_1 \times \cdots \times \mathbf{A_n})_I = A_{1,I} \times \cdots \times A_{n,I}$;
    2. $(\mathbf{PA})_I = \mathcal{P}A_I = \Omega^{A_I}$, where Ω is the truth value object of $\mathcal{E}$;
    3. $1_I = 1$, the terminal object of $\mathcal{E}$.
- each function symbol $\mathbf{f}\colon \mathbf{A} \to \mathbf{B}$ is assigned an $\mathcal{E}$-arrow $f_I\colon A_I \to B_I$.

The interpretation is extended to terms of $\mathcal{L}$ in the following way. If $\tau\colon \mathbf{B}$, *i.e.* τ is of type $\mathbf{B}$, we write $\mathbf{x}$ for any sequence $(x_1, \ldots, x_n)$ of variables containing all variables of τ. Such sequences are called *adequate sequences*. Then we have the following recursive definition of an $\mathcal{E}$-arrow

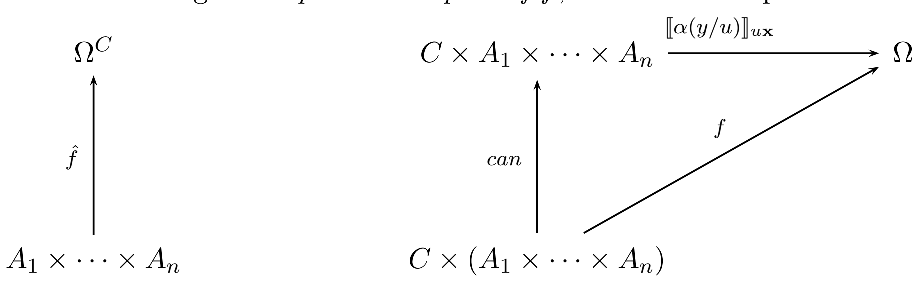
$$[\![\tau]\!]_{\mathbf{x}}\colon A_1 \times \cdots \times A_n \to B :$$

$$\begin{aligned}
&[\![\bullet]\!]_{\mathbf{x}} = A_1 \times \cdots \times A_n \to 1 \text{ (the unique such arrow in } \mathcal{E})\\
&[\![x_i]\!]_{\mathbf{x}} = p_i\colon A_1 \times \cdots \times A_n \to A_i \text{ (the projection arrow of the product)}\\
&[\![\mathbf{f}(\tau)]\!]_{\mathbf{x}} = f_I \circ [\![\tau]\!]_{\mathbf{x}}\\
&[\![\langle \tau_1, \ldots, \tau_n\rangle]\!]_{\mathbf{x}} = \langle [\![\tau_1]\!]_{\mathbf{x}}, \ldots, [\![\tau_n]\!]_{\mathbf{x}}\rangle\\
&[\![(\tau)_i]\!]_{\mathbf{x}} = p_i \circ [\![\tau]\!]_{\mathbf{x}}\\
&[\![\{y \mid \alpha\}]\!]_{\mathbf{x}} = ([\![\alpha(y/u)]\!]_{u\mathbf{x}} \circ can)^{\wedge}
\end{aligned}$$

The last line requires unpacking. u is not one of the x_i, but rather is free for y in α, y is of type $\mathbf{C}$ making $\mathbf{B}$ of type $\mathbf{PC}$, and *can* is the canonical isomorphism

$$C \times (A_1 \times \cdots \times A_n) \cong C \times A_1 \times \cdots \times A_n.$$

The $^{\wedge}$ indicates the taking of the *power transpose of f*, which is the exponential adjoint of f:



Next we have that

$$[\![\sigma = \tau]\!]_{\mathbf{x}} = eq_C \circ [\![\langle \sigma, \tau\rangle]\!]_{\mathbf{x}},$$

where eq_C is the the character $\chi_{\delta_C}\colon C \times C \to \Omega$ of the diagonal arrow $\delta_C\colon C \rightarrowtail C \times C$. Finally,

$$[\![\sigma \in \tau]\!]_{\mathbf{x}} = e_C \circ [\![\langle \sigma, \tau\rangle]\!]_{\mathbf{x}},$$

where $e_C\colon \Omega^C \times C \to \Omega$ is the evaluation arrow of the power object Ω^C.

Now, in the special case where τ is a closed term (of type $\mathbf{B}$), then $\mathbf{x}$ can be taken to be the empty set $\varnothing$. In such a case we denote $[\![\tau]\!]_\varnothing$ by $[\![\tau]\!]$. In this case $[\![\tau]\!]$ is an $\mathcal{E}$-element $1 \to B$ of B, and if τ is a closed set-like term $\{y \mid \alpha\}$ then $[\![\{y \mid \alpha\}]\!]$ is an $\mathcal{E}$-element of $PC = \Omega^C$, which corresponds via the taking of the power transpose to a subobject of C. Then the arrow $[\![\alpha]\!]_y$ is precisely the classifying arrow of this subobject of C.

We may now define the notion of the validity of a formula in a topos. Let I be an interpretation of a local language $\mathcal{L}$ in a topos $\mathcal{E}$. If $\Gamma = \{\alpha_1, \dots, \alpha_m\}$ is a finite set of formulas we write $[\![\Gamma]\!]_{I,\mathbf{x}}$ for

$$\begin{cases} [\![\alpha_1]\!]_{I,\mathbf{x}} \cap \cdots \cap [\![\alpha_m]\!]_{I,\mathbf{x}} & m > 0 \\ \top & m = 0. \end{cases}$$

If β is a formula and $\mathbf{x} = (\mathbf{x_1}, \dots, \mathbf{x_n})$ is a list of all the free variables in $\Gamma \cup \{\beta\}$, then we write

$$\Gamma \vDash_I \beta \quad \text{and} \quad \Gamma \vDash_{\mathcal{E}} \beta$$

if

$$[\![\Gamma]\!]_{I,\mathbf{x}} \leq [\![\beta]\!]_{I,\mathbf{x}},$$

where the latter expression means that $[\![\Gamma]\!]_{I,\mathbf{x}} \in [\![\beta]\!]_{I,\mathbf{x}}$ in the sense that there is an arrow f such that $[\![\Gamma]\!]_{I,\mathbf{x}} = [\![\beta]\!]_{I,\mathbf{x}} \circ f$ (this relation turns the collections of arrows $\mathcal{E}(A, \Omega)$ into a lattice). A formula β of a local language $\mathcal{L}$ is *valid* in a topos $\mathcal{E}$ under the interpretation I if $\Gamma \vDash_I \beta$. In such a case we may also say that the sequent Γ: β is valid under an interpretation I in $\mathcal{E}$. If S is a local set theory, then I is a *model* of S if every sequent of S is valid under I. We write

$$\Gamma \vDash \alpha \quad \text{for} \quad \Gamma \vDash_I \alpha \text{ for every interpretation } I$$

and we write

$$\Gamma \vDash_S \alpha \quad \text{for} \quad \Gamma \vDash_I \alpha \text{ for every model } I \text{ of } S.$$

There are several important facts about interpretations of local languages in toposes. First of all, it can be shown that the basic axioms and rules of inference of any local set theory are valid under every interpretation. It can also be shown that interpretations of local languages and local set theories are *sound*, *i.e.*

$$\Gamma \vdash \alpha \Longrightarrow \Gamma \vDash \alpha$$

and

$$\Gamma \vdash_S \alpha \Longrightarrow \Gamma \vDash_S \alpha.$$

For a proof see Bell (2008, 97 *ff.*). Furthermore, it can be shown that interpretations of local languages and local set theories are *complete*, *i.e.*

$$\Gamma \vDash \alpha \Longrightarrow \Gamma \vdash \alpha$$

and

$$\Gamma \vDash_S \alpha \Longrightarrow \Gamma \vdash_S \alpha.$$

For a proof see Bell (2008, 103-105).

Another important result is that every topos is equivalent to the topos of sets of some local set theory. Thus, the notion of a local set theory really is an appropriate notion for interpreting logic in a topos and interpreting a topos as a generalized category of sets. This result is established by showing that every topos generates a local language $\mathcal{L}$, called the *internal language* of the topos. Given a topos $\mathcal{E}$, the internal local language, or *theory* is denoted $Th(\mathcal{E})$. The topos of sets $\mathcal{C}(S)$ generated by a local set theory $\mathcal{L}$ is called a *linguistic topos*. It can then be shown that the topos of sets $\mathcal{C}(Th(\mathcal{E}))$ generated by the internal language $Th(\mathcal{E})$ of a topos $\mathcal{E}$ is equivalent, in the technical sense of an equivalence of categories, to the topos $\mathcal{E}$ itself, *i.e.*

$$\mathcal{E} \cong \mathcal{C}(Th(\mathcal{E})).$$

Another interesting fact about local set theory is that a version of Tarski's theorem on the undefinability of truth and Gödel's first and second incompleteness theorems can be stated and proven in any local set theory. Not only this, the proofs of these theorems is considerably simpler than in the context of classical first order logic. For details and proofs see Bell (2005, 321-324). Also, like the characterization of the category **Set** of sets considered above, it can be shown that **Set** is equivalent to a particular kind of local set theory. For details and a proof see Bell (2005, 325-327).

7 Number Systems, Arithmetic and Natural Numbers Objects

As we saw in a previous section, the category **Cat** of sets can be characterized as a well-pointed topos with a natural numbers object such that epics split. We now consider how to define the 'set' of natural numbers in a topos. In set theory the axiom of infinity guarantees the existence of the infinite set ω, the set of finite ordinals. We wish to define an object of a topos that corresponds to ω in **Set**.

ω has a designated element $0 = \varnothing$ and the rest of the natural numbers are generated from it. 1 is $\{0\}$, which is generated from 0 as $0 \cup \{0\}$. 2 is $\{0, 1\} = \{0, \{0\}\}$, which is generated from 1 as $1 \cup \{1\}$. Similarly, for any finite ordinal n, $n + 1 =_{def} n \cup \{n\}$. This generates the set ω of natural numbers recursively. This also generates a function s: $\omega \to \omega$ on ω defined such that $s(n) = n \cup \{n\} = n + 1$ called the *successor function*. The diagram corresponding to this is the following:

$$1 \xrightarrow{0} \omega \xrightarrow{s} \omega.$$

Lawvere noticed that this diagram has a sort of "co-universal" property, in the following way. (Goldblatt, 2006, 301) Given any diagram of the same type, *i.e.* a diagram of the form

$$1 \xrightarrow{x} A \xrightarrow{f} A,$$

it factors uniquely through the diagram involving the successor function above.

To see this observe that we can generate a sequence $\langle x(0), f(x(0)), f(f(x(0))), \ldots \rangle$ of elements of A by repeatedly applying the function f to $x(0)$. Since sequences are properly thought of as functions, we obtain a function h: $\omega \to A$ defined such that

$$h(0) = x(0)$$

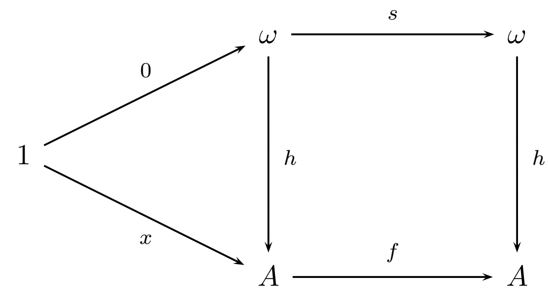
and

$$h(n + 1) = f(h(n)),$$

which in terms of the successor function says that

$$h \circ s(n) = f \circ h(n).$$

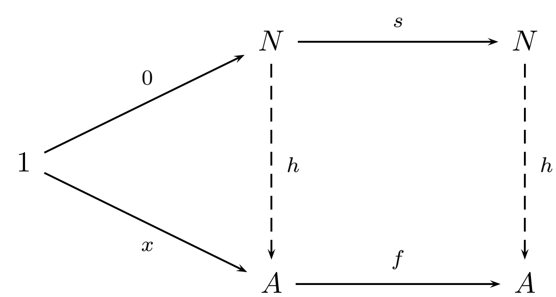
These conditions on h express the commutativity of the following diagram:



Thus, we see the sense in which f factors through s. In order for this diagram to commute we must have that $h(0) = x$ and that $h(1) = h(s(0)) = f(h(0)) = f(x)$. But from this latter relation, commutativity requires that $h(2) = h(s(1)) = h(s(s(0))) = f(f(h(0))) = f(f(x)) = f(h(1))$. By induction we see that commutativity requires that the defining conditions for the function h must hold, *i.e.* that the factorization is unique.

This method of definition of functions is called definition by *simple recursion*. To enable definition by simple recursion in a category we may introduce the following axiom, which is the axiom for a *natural numbers object*:

Axiom 3 (Natural Numbers Object (NNO)) There exists a *natural numbers object*, *i.e.* an object N with arrows $1 \xrightarrow{0} N \xrightarrow{s} N$ such that for any object A and arrows $1 \xrightarrow{x} A \xrightarrow{f} A$ there is a unique arrow $h\colon N \to A$ making the following diagram commute:



This characterization of a natural numbers object is called the *Peano-Lawvere axiom*. It follows from the definition of a natural numbers object that if a category has a natural numbers object then it is unique up to isomorphism, *i.e.* the arrows between them guaranteed to exist by the definition are iso.

In the context of local set theories, natural numbers objects $1 \xrightarrow{0} N \xrightarrow{s} N$ are characterized axiomatically in the following way:

$$\vdash_S \neg(s(n) = 0);$$

$$s(n) = s(m) \vdash_S m = n;$$

$$0 \in u, \forall(n \in u \to s(n) \in u) \vdash_S \forall n.n \in u;$$

where m and n are variables of type N, the variable u is of type PN. These are called the *Peano axioms*. The last axiom is seen to be the axiom schema of induction. It, naturally, is what makes natural numbers objects support induction, since it says that the only subobject of N including 0 and is closed under succession is N itself. The Peano-Lawvere definition for natural numbers objects given above is also called the *simple recursion principle*. It can be shown that a diagram $1 \xrightarrow{0} N \xrightarrow{s} N$ satisfies the simple recursion principle (Peano-Lawvere axiom) iff it satisfies the Peano axioms, *i.e.* the two characterizations of a natural numbers object are equivalent.

Recalling that the existence of ω in **Set** is a result of the axiom of infinity, one might wonder if such a connection may be made in toposes. In fact it can. It can be shown that the existence in a topos $\mathcal{E}$ of a natural numbers object is equivalent to the existence of an $\mathcal{E}$-object A and an iso arrow f such that $A+1 \xrightarrow{f} A$. If we define an object of a topos to be *infinite* if $A + 1 \simeq A$, then it follows that a topos contains a natural numbers object iff it contains an infinite object. Thus, the Peano-Lawvere axiom really is like an axiom of infinity. Recalling that **Set** is characterized categorically as a well-pointed topos with a natural numbers object such that epics split, we see that the addition of the existence of a natural numbers object to a well-pointed topos satisfying ES amounts to the addition of an axiom of infinity.

An important theorem that can be proved in any topos with a natural numbers object is the *primitive recursion theorem*. This theorem licences the definition of functions by primitive recursion in any topos with a natural numbers object, thereby enabling the definition of the operations required to set up a system of arithmetic. This includes an order relation on N as well as addition and multiplication operations. Thus, any topos with a natural numbers object includes a system of arithmetic.

This can be taken further, since toposes with a natural numbers object admit forms of the usual construction of the integers, rationals and Cauchy and Dedikind reals. Interestingly, the construction of the reals in a topos by the Cauchy and Dedikind procedures do not in general lead to isomorphic results. Given a construction of the rationals Q, the object Q^N is an object of sequences of rationals. In the internal language of a topos it is then possible to define Cauchy seqences and the required equivalence relation in order to produce the object R_c of Cauchy reals. Since a Dedikind cut is a suitable pair of subobject of the rationals, the Dedikind reals R_d are a subobject of $\Omega^Q \times \Omega^Q$. For the details of these constructions see Johnstone (1977). It is not true in general that R_d is (conditionally) order-complete.[14] In fact, it was shown by Johnstone that R_d is conditionally complete iff the local set theory S one is working in satisfies the logical rule

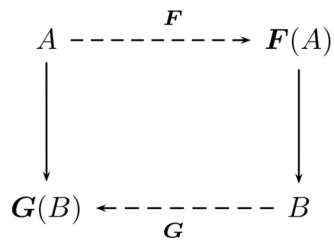
$$\vdash_S \forall\omega(\neg\omega \vee \neg\neg\omega),$$

showing an interesting connection between mathematics and logic in local set theories.

[14] A partially ordered set is *conditionally complete* if every non-empty set with an upper bound has a least upper bound.

A Adjoint Situations

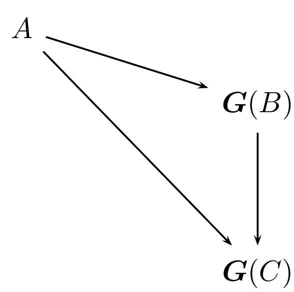
As the definition of adjoint in the first section indicates, an adjunction involves a pair of functors $\boldsymbol{F}\colon \mathbf{C} \to \mathbf{D}$ and $\boldsymbol{G}\colon \mathbf{D} \to \mathbf{C}$ between categories $\mathbf{C}$ and $\mathbf{D}$ which establishes a certain kind of correspondence of arrows of the two categories. Let A be a $\mathbf{C}$ object and a B a $\mathbf{D}$ object. In an adjoint situation, if there is an arrow from A to $\boldsymbol{G}(B)$ in $\mathbf{C}$, then there is a corresponding arrow $\boldsymbol{F}(A)$ to B in $\mathbf{D}$. Similarly, if there is an arrow from $\boldsymbol{F}(A)$ to B, there is a corresponding arrow from A to $\boldsymbol{G}(B)$. Thus, we have a correspondence of the form



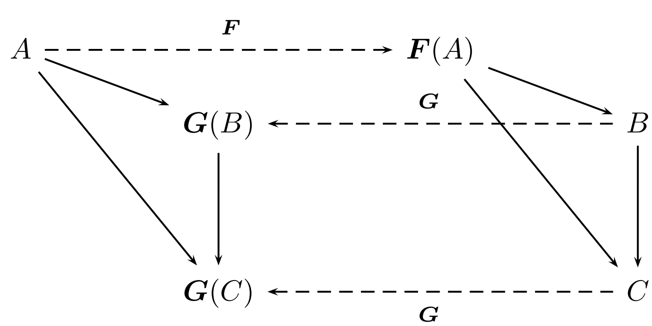
and a bijection θ_{AB} between the collections of arrows:

$$\mathbf{C}(A, \boldsymbol{G}(B)) \cong \mathbf{D}(\boldsymbol{F}(A), B).$$

You will recall that the definition of an adjunction also includes a pair of natural transformations. This fits in in the present picture in terms of the correspondence between the hom-sets $\mathbf{C}(A, \boldsymbol{G}(B))$ and $\mathbf{D}(\boldsymbol{F}(A), B)$ being *natural in* A *and* B, which is to say that the correspondence θ_{AB} preserves the structure of the categories as the selection of objects A and B from $\mathbf{C}$ and $\mathbf{D}$, respectively, varies. For example, suppose we consider pairs $\langle A, B\rangle$ and $\langle A, C\rangle$ in $\mathbf{C} \times \mathbf{D}$ such that the following is a commutative diagram in $\mathbf{C}$:

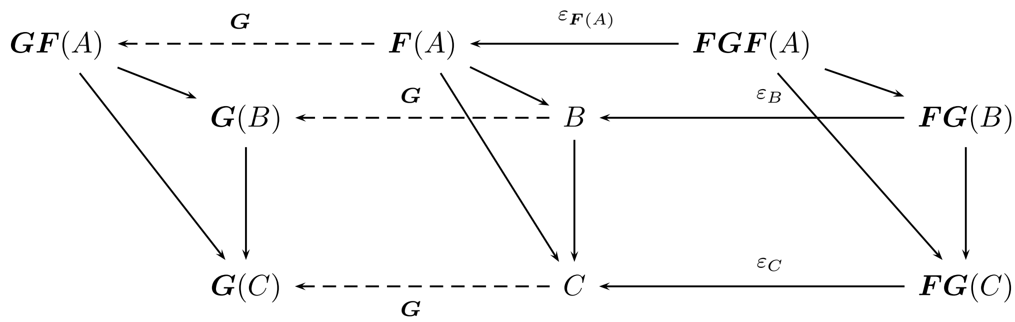


Then, there is a corresponding commutative triangle in $\mathbf{D}$ such that θ_{AB} and θ_{AC} relates the corresponding arrows:

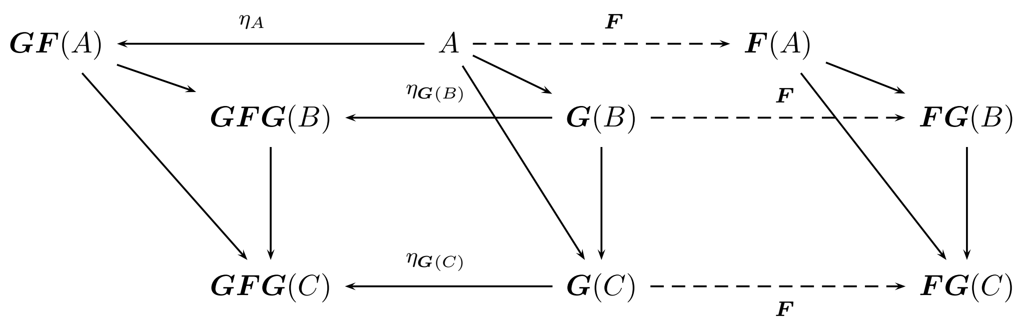


The correspondences θ_{AB} generate the components of a natural transformation between the two functors. To see how this works, consider how the diagram on the right in the previous figure gets mapped into $\mathbf{C}$ by $\boldsymbol{G}$ and then how this diagram gets mapped back into $\mathbf{D}$

by $\boldsymbol{F}$. We have the following:



The arrows labelled $\varepsilon_{\boldsymbol{F}(A)}$, ε_B and ε_C are components of the natural transformation $\varepsilon\colon\ \boldsymbol{FG} \to \mathbf{1}_{\boldsymbol{D}}$ given in the definition of adjunction in the first section. Similarly, if we consider the same for the diagram on the left being mapped into $\mathbf{D}$ by $\boldsymbol{F}$ and then how this diagram gets mapped back into $\mathbf{C}$ by $\boldsymbol{G}$ then we obtain:



The arrow labelled η_A, $\eta_{\boldsymbol{G}(B)}$ and $\eta_{\boldsymbol{G}(C)}$ are components of the natural transformation $\eta\colon\ \mathbf{1}_{\boldsymbol{C}} \to \boldsymbol{GF}$ given in the definition of adjunction in the first section.

Given such a situation, a triple $\langle \boldsymbol{F}, \boldsymbol{G}, \theta \rangle$, which is an equivalent definition of an *adjunction* between $\mathbf{C}$ and $\mathbf{D}$, $\boldsymbol{F}$ is said to be *left adjoint* to $\boldsymbol{G}$ (in such a case we say $\boldsymbol{F}$ *has a left adjoint*), which is denoted $\boldsymbol{F} \dashv \boldsymbol{G}$, and $\boldsymbol{G}$ is said to be *right adjoint* to $\boldsymbol{F}$ (in such a case we say $\boldsymbol{G}$ *has a right adjoint*), which is denoted $\boldsymbol{G} \vdash \boldsymbol{F}$. The correspondence is sometimes denoted as

$$\frac{\boldsymbol{F}(A) \to B}{A \to \boldsymbol{G}(B)},$$

with $\boldsymbol{F}(A)$ being on the left and $\boldsymbol{G}(B)$ being on the right, corresponding the which is the right and left adjoint. The existence of a right or a left adjoint to a functor has important implications for the structures preserved by the functor. For example, if $\boldsymbol{F} \dashv \boldsymbol{G}$ then $\boldsymbol{G}$ preserves limits, *i.e.* maps limits of $\mathbf{D}$ to limits of $\mathbf{C}$, and $\boldsymbol{F}$ preserves colimits.

Adjunctions are ubiquitous in mathematics. The thesis of Mac Lane (1978) is that "a systematic use of ... adjunctions illuminates and clarifies the [areas of mathematics where adjunctions arise]." (107) In the words of Goldblatt (2006), "the isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas." (438) Let us give a few examples.

For a simple example of an adjoint, consider the one object category $\mathbf{1}$ and another category $\mathbf{D}$. Then, there is a unique functor $\mathbf{G}$ from $\mathbf{D}$ to $\mathbf{1}$. Thus, if $\boldsymbol{F}\colon\ \mathbf{1} \to \mathbf{D}$ is a functor such that

$\boldsymbol{F} \dashv \boldsymbol{G}$, *i.e.* $\boldsymbol{F}$ is left adjoint to $\mathbf{G}$, then for any object B of $\mathbf{D}$, there is a correspondence

$$\frac{\boldsymbol{F}(0) \to B}{0 \to \boldsymbol{G}(B)}.$$

Now, since there is a unique arrow $0 \to \boldsymbol{G}(B)$ since $\mathbf{1}$ has only one object, there is a unique arrow $\boldsymbol{F}(0) \to B$ in $\mathbf{D}$, which is to say that $\boldsymbol{F}(0)$ is an initial object!

Let $\mathcal{S} = \mathbf{Set}$. Another example involves the *diagonal functor* $\Delta\colon \mathcal{S} \to \mathcal{S} \times \mathcal{S}$, defined by $X \mapsto \langle X, X\rangle$ (and $f \mapsto \langle f, f\rangle$). Δ has a left adjoint as a result of the correspondence between pairs of functions $C \to X$ and $C \to Y$ and mappings $C \to X + Y$ to the coproduct $X + Y$, *i.e.* we have a correspondence

$$\frac{\Delta(C) \to \langle X, Y\rangle}{C \to X + Y}.$$

Thus, the functor $\amalg\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ right adjoint to Δ is the functor that sends a pair $\langle X, Y\rangle$ to is coproduct $X + Y$, *i.e.* $\amalg(\langle X, Y\rangle) = X + Y$.

Δ also has a right adjoint as a result of the correspondence between pairs of functions $X \to C$ and $Y \to C$ and mappings $X \times Y \to C$ to the product $X \times Y$, *i.e.* we have a correspondence

$$\frac{X \times Y \to C}{\langle X, Y\rangle \to \Delta(C)}.$$

Thus, the functor $\Pi\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ left adjoint to Δ is the functor that sends a pair $\langle X, Y\rangle$ to the product $X \times Y$, *i.e.* $\Pi(\langle X, Y\rangle) = X \times Y$. Thus, we have that

$$\Pi \dashv \Delta \dashv \amalg.$$

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